Math 322 lecture for Wednesday, Week 6

## NONLINEAR SYSTEMS

Let $E \subseteq \mathbb{R}^n$ be an open subset of $\mathbb{R}^n$ and let $C(E)$ denote the vector space of continuous functions of the form $E \to \mathbb{R}^n$. Given $f \in C(E)$, we are now interested in solutions to the differential equation

$$x' = f(x). \tag{1}$$

The function $f$ is a vector field in $\mathbb{R}^n$ defined on $E$. We have just finished studying the linear case of this problem, i.e., in which $f(x) = Ax$ for some $A \in M_n(\mathbb{R}^n)$ and are now particularly interested in the case where $f$ is no longer a linear function.

A *solution* to equation (1) on an interval $I$ is a function $x\colon I \to E \subseteq \mathbb{R}^n$ such that

$$x'(t) = f(x(t))$$

for all $t \in I$. Given $t_0 \in I$ with $x(t_0) = x_0 \in E$, we say the solution satisfies the *initial value problem*

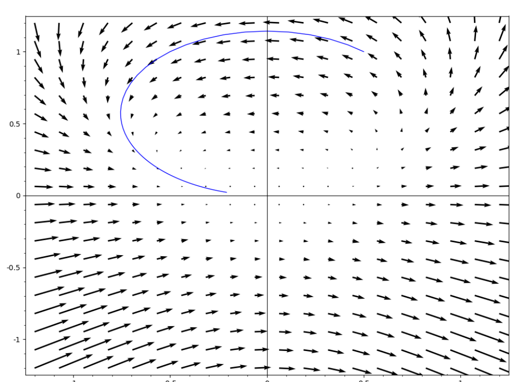
$$\begin{aligned} x' &= f(x) \\ x(t_0) &= x_0 \end{aligned}$$

on $I$.

**Example.** Consider the (non-linear) system

$$\begin{aligned} x' &= x^2 - y \\ y' &= xy \end{aligned}$$

with initial value $(x(0), y(0)) = (0.5, 1)$. So in this case, the relevant vector field is $f(x, y) = (x^2 - y, xy)$. Here is a plot of the vector field and the solution to the initial value problem:



Note that this system displays behavior one would not see in the linear case.

The systems we are studying are called *autonomous* since $f$ is a function of $x \in \mathbb{R}^n$ and not $t$. However, a *nonautonomous* system

$$x' = g(x, t)$$

can be converted to an autonomous system by letting $x_{n+1} := t$ and $x'_{n+1} = 1$.

**Goals.** Our first main goal is to find conditions under which the initial value problem for equation (1) has a unique solution. After that, we'll discuss how solutions change if $f$ changes a small amount and discuss the size of the interval on which a solution exists.

**New behavior.** In the linear case, $x' = Ax$ and $x(0) = x_0$, we saw that there is always a unique solution. That's no longer generally true in the nonlinear case. For instance, the following initial value problem

$$\begin{aligned} x' &= 3x^{2/3} \\ x(0) &= 0 \end{aligned}$$

has two solutions: $x(t) = 0$ and $x(t) = t^3$. We'll see that the source of non-uniqueness here is that $f(x) = 3x^{2/3}$ is not continuously differentiable: $f'(x) = 2x^{-1/3}$, which is not continuous at 0.

Even if $f'$ is continuous everywhere, the solution may only exist on subintervals of the real line, again unlike the linear situation. For example, consider the system

$$\begin{aligned} x' &= x^2 \\ x(0) &= 1. \end{aligned}$$

The solution is

$$x(t) = \frac{1}{1-t}$$

but is only defined on the interval $(-\infty, 1)$. The solution blows up as $t \to 1^-$.

**Key idea.** We have solved the initial value problem for equation (1) if we can find a continuous function $x(t)$ satisfying

$$x(t) = x_0 + \int_{s=0}^{t} f(x(s))\, ds$$

for all $t \in [-a, a]$ for some $a > 0$.

Check: First, by the fundamental theorem of calculus

$$x'(t) = (x_0)' + \left(\int_{s=0}^{t} f(x(s))\,ds\right)' = 0 + f(x(t)) = f(x(t)).$$

Next,

$$x(0) = x_0 + \int_{s=0}^{0} f(x(s))\,ds = x_0.$$

The *method of successive approximations* attempts to create a sequence of functions $(u_k(t))_{k\geq 0}$ converging to a solution:

$$\begin{aligned} u_0(t) &:= x_0 \\ u_{k+1}(t) &:= x_0 + \int_{s=0}^{t} f(u_k(s))\,ds, \quad \text{for } k \geq 0. \end{aligned}$$

**Example.** Consider the initial value problem

$$x' = xt, \quad x(0) = 1.$$

This is an autonomous equation, so we first convert it to a nonautonomous system by letting $x_1 = x$ and $x_2 = t$. The system becomes

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} x_1 x_2 \\ 1 \end{pmatrix} =: f(x_1, x_2)$$

with initial condition $x_1(0) = x(0) = 1$ and $x_2(0) = 0$ (since $x_2 = t$).
Apply the method of successive approximations starting with

$$u_0(t) = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

We get

$$\begin{aligned} u_1(t) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_{s=0}^{t} f(u_0(s))\,ds \\ &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_{s=0}^{t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} ds \\ &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} c \\ s \end{pmatrix}\Bigg|_{s=0}^{t} \\ &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ t \end{pmatrix}. \end{aligned}$$

Next,

$$
\begin{aligned}
u_2(t) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_{s=0}^{t} f(u_1(s))\, ds \\
&= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_{s=0}^{t} f(1, s)\, ds \\
&= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_{s=0}^{t} \begin{pmatrix} s \\ 1 \end{pmatrix} ds \\
&= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} s^2/2 \\ s \end{pmatrix}\Bigg|_{s=0}^{t} \\
&= \begin{pmatrix} 1 + t^2/2 \\ t \end{pmatrix}.
\end{aligned}
$$

Next,

$$
\begin{aligned}
u_3(t) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_{s=0}^{t} f(u_2(s))\, ds \\
&= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_{s=0}^{t} f(1 + s^2/2, s)\, ds \\
&= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_{s=0}^{t} \begin{pmatrix} s + s^3/2 \\ 1 \end{pmatrix} ds \\
&= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} s^2/2 + s^2/(2 \cdot 4) \\ s \end{pmatrix}\Bigg|_{s=0}^{t} \\
&= \begin{pmatrix} 1 + t^2/2 + t^4/(2 \cdot 4) \\ t \end{pmatrix}.
\end{aligned}
$$

Similarly,

$$
u_4 = \begin{pmatrix} 1 + t^2/2 + t^4/(2 \cdot 4) + t^6/(2 \cdot 4 \cdot 6) \\ t \end{pmatrix},
$$

and so on. Recall that $x_1 = x$, and $x$ is the function we are trying to find. Thus, we are interested in the limit of the first components of the $u_k$. The method of successive approximations is delivering

$$
\begin{aligned}
x(t) &= 1 + \frac{t^2}{2} + \frac{t^4}{2\cdot 4} + \frac{t^6}{2\cdot 4\cdot 6} + \frac{t^8}{2\cdot 4\cdot 6\cdot 8} + \dots \\
&= 1 + \frac{t^2}{2} + \frac{t^4}{(1\cdot 2)2^2} + \frac{t^6}{(1\cdot 2\cdot 3)2^3} + \frac{t^8}{(1\cdot 2\cdot 3\cdot 4)2^3} + \dots \\
&= 1 + \frac{t^2}{2} + \frac{1}{2!}\left(\frac{t^2}{2}\right)^2 + \frac{1}{3!}\left(\frac{t^2}{2}\right)^3 + \frac{1}{4!}\left(\frac{t^2}{2}\right)^4 + \dots \\
&= e^{t^2/2},
\end{aligned}
$$

which converges, and it's easy to check that it satisfies the original initial value problem:

$$x'(t) = \left(e^{t^2/2}\right)' = te^{t^2/2} = x(t)t,$$

and $x(0) = 1$.

Of course, we could have solved the equation through separation of variables:

$$x' = xt \quad \Rightarrow \quad \int \frac{dx}{x} = \int t\,dt \quad \Rightarrow \quad \ln(x) = t^2/2 + c.$$

Then $x(0) = 1$ implies $c = 0$. Exponentiate:

$$\ln(x) = t^2/2 \quad \Rightarrow \quad x = e^{t^2/2}.$$

**Fixed points.** Consider the operator on functions, $u \to T(u)$ given by

$$T(u)(t) := x_0 + \int_{s=0}^{t} f(u(s))\,ds$$

In the case we just considered, with $x_0 = (1,0)$ and $f(x_1, x_2) = (x_1x_2, 1)$, the method of successive iterations produced the function $u(t) = (e^{t^2/2}, t)$. This function $u$ is a *fixed point* for the operator $T$:

$$\begin{aligned}
T(u(t)) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_{s=0}^{t} f(u(s))\, ds \\
&= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_{s=0}^{t} f(e^{s^2/2}, s)\, ds \\
&= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_{s=0}^{t} \begin{pmatrix} se^{s^2/2} \\ 1 \end{pmatrix} ds \\
&= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} e^{s^2/2} \\ s \end{pmatrix} \Bigg|_{s=0}^{t} \\
&= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} e^{t^2/2} - 1 \\ t \end{pmatrix} \\
&= \begin{pmatrix} e^{t^2/2} \\ t \end{pmatrix} \\
&= u(t).
\end{aligned}$$

**Next step.** We have seen that the method of successive approximations amounts to iterating a operator on a space of functions and converging to a fixed point for that operator. Our next step is to consider this situation a little more generally. Let $X$ be a space in which convergence makes sense, and consider a mapping $T\colon X \to X$. We would like to know conditions under which iterates of a point $x_0 \in X$ under $T$ will converge to a point $\tilde{x} \in X$ that is fixed under $T$, i.e., such that $T(\tilde{x}) = \tilde{x}$.