Math 322 lecture for Friday, Week 6

THE CONTRACTION MAPPING PRINCIPLE

Let $(V, \| \ \|)$ be a normed vector space over $F = \mathbb{R}$ or $\mathbb{C}$. Recall this means that for all $v, w \in V$ and $\alpha \in F$,

(a) $\|v\| \geq 0$ with equality if and only if $v = 0$;

(b) $\|\alpha v\| = |\alpha| \|v\|$;

(c) $\|v + w\| \leq \|v\| + \|w\|$.

If every Cauchy sequence in V converges (in V), then we say V is *complete*, and in that case $(V, \| \ \|)$ is called a *Banach space*. We have already used the fact that $\mathbb{R}^n$ and $\mathbb{C}^n$ are Banach spaces, for example, when considering the convergence of e^{At}. We will soon need to consider a Banach space whose elements consist of potential solutions to systems of differential equations.

Definition. Let $(V, \| \ \|)$ be a Banach space, and let $X \subseteq V$. Let $T\colon X \to X$.

(a) A point $u \in X$ is a *fixed point* for T if $T(u) = u$.

(b) The function T is a *contraction mapping* if there is a constant $c \in [0, 1) \subset \mathbb{R}$ such that
$$\|T(u) - T(v)\| \leq c\|u - v\|$$
for all $u, v \in X$.

Theorem. Let $(V, \| \ \|)$ be a Banach space, and let $X \subseteq V$ be a closed subset of V (hence, it contains all of its limit points). Suppose that $T\colon X \to X$ is a contraction mapping and fix a constant $c \in [0, 1) \subset \mathbb{R}$ so that
$$\|T(u) - T(v)\| \leq c\|u - v\|$$
for all $u, v \in X$. Then T has a unique fixed point $\tilde{u} \in X$. Let $u_0 \in X$ and consider the sequence of iterates
$$u_0, T(u_0), T^2(u_0), T^3(u_0), \ldots$$
(For example, $T^3(u_0) = T(T(T(u_0)))$.) We have, for all $m \geq 0$,
$$\|\tilde{u} - T^m(u_0)\| \leq \frac{c^m}{1 - c}\|T(u_0) - u_0\|.$$
In particular, the sequence of iterates converges to the fixed point, $\tilde{u}$.

Proof. We first show uniqueness. Suppose that $T(u) = u$ and $T(v) = v$. We have

$$\|u - v\| = \|T(u) - T(v)\| \leq c\|u - v\|,$$

which implies

$$(1 - c)\|u - v\| \leq 0.$$

Since $1 - c \geq 0$, it follows that $\|u - v\| = 0$, and hence, $u = v$.

Now take $u_0 \in X$, and define $u_{k+1} := T(u_k)$ for $k \geq 0$. Thus, $u_k = T^k(u_0)$ for all $k \geq 0$. For all pairs of natural numbers $m \leq n$,

$$\begin{aligned}
\|u_n - u_m\| &= \|T(u_{n-1}) - T(u_{m-1})\| \\
&\leq c\|u_{n-1} - u_{m-1}\| \\
&= c\|T(u_{n-2}) - T(u_{m-2})\| \\
&\leq c^2\|u_{n-2} - u_{m-2}\| \\
&\ \vdots \\
&\leq c^m\|u_{n-m} - u_0\| \\
&= c^m\|(u_1 - u_0) + (u_2 - u_1) + (u_3 - u_2) \cdots + (u_{n-m} - u_{n-m-1})\| \\
&\leq c^m\left(\|u_1 - u_0\| + \|u_2 - u_1\| + \|u_3 - u_2\| + \cdots + \|u_{n-m} - u_{n-m-1}\|\right) \\
&\leq c^m\left(\|u_1 - u_0\| + c\|u_1 - u_0\| + c^2\|u_1 - u_0\| + \cdots + c^{n-m-1}\|u_1 - u_0\|\right) \\
&= c^m\|u_1 - u_0\|\left(1 + c + c^2 + \cdots + c^{n-m-1}\right) \\
&= c^m\frac{1 - c^{n-m}}{1 - c}\|u_1 - u_0\| \\
&\leq \frac{c^m}{1 - c}\|u_1 - u_0\|.
\end{aligned}$$

Given any $\varepsilon > 0$, we then see that by choosing N sufficiently large, it follows that if $m, n \geq N$, then $\|u_n - u_m\| < \varepsilon$. So the sequence of iterates, $(u_k)_{k\geq 0}$ is Cauchy. Since V is a Banach space, the sequence converges to some $\tilde{u}$, and since X is closed, $\tilde{u} \in X$. Since T is a contraction mapping, it's continuous (exercise), and therefore commutes with limits:

$$\lim_{k\to\infty} T(u_k) = T(\lim_{k\to\infty} u_k) = T(\tilde{u}).$$

On the other hand, by definition of the u_k, we have

$$\lim_{k\to\infty} T(u_k) = \lim_{k\to\infty} u_{k+1} = \lim_{k\to\infty} u_k = \tilde{u}.$$

This shows $T(\tilde{u}) = \tilde{u}$, i.e., $\tilde{u}$ is the unique fixed point of T.

Finally, in our calculation above, we saw that for all $m \leq n$,

$$\|u_n - u_m\| = \|T^n(u_0) - T^m(u_0)\| \leq \frac{c^m}{1-c}\|u_1 - u_0\| \leq \frac{c^m}{1-c}\|T(u_0) - u_0\|.$$

Since the norm function and T are continuous, they both commute with limits. Therefore, taking the limit as $n \to \infty$ on both sides of the above inequality yields

$$\|\tilde{u} - T^m(u_0)\| \leq \frac{c^m}{1-c}\|T(u_0) - u_0\|.$$

$\square$

Method of successive approximations. We are interested in applying the contraction mapping principle to the operator

$$T(u) = x_0 + \int_{s=0}^{t} f(u(s))\,ds,$$

discussed in the previous lecture. So we need to find the appropriate Banach space and find conditions under which T is a contraction mapping.

Definition. If $I \subset \mathbb{R}$ is a closed bounded interval, let $C(I)$ denote the $\mathbb{R}$-vector space of continuous functions on $I \to \mathbb{R}^n$ (where n is fixed). For each $u \in C(I)$, define

$$\|u\| := \sup_{t\in I} |u(t)| = \max_{t\in I} |u(t)|.$$

(The last equality is due to the fact that the continuous image of a compact set is compact—a generalization of the extreme value theorem of one-variable calculus.) Geometrically, $\|u\|$ is the maximum distance from the origin reached by $u(t)$.

Proposition. $(C(I), \|\ \|)$ is a Banach space.

Proof. Math 321. $\square$

Thus, the method of successive approximations is an operator

$$T\colon C(I) \to C(I)$$

on the Banach space of continuous functions on I. Under what conditions is it a contraction mapping? We have

$$|(Tu)(t) - (Tv)(t)| = \left| \left(x_0 + \int_{s=0}^{t} f(u(s))\,ds\right) - \left(x_0 + \int_{s=0}^{t} f(v(s))\,ds\right) \right|$$

$$= \left| \int_{s=0}^{t} f(u(s)) - f(v(s))\, ds \right|$$

$$\leq \int_{s=0}^{t} |f(u(s)) - f(v(s))|\; ds$$

$$\leq t \max_{s\in[0,t]} \{|f(u(s)) - f(v(s))|\}.$$

From this, we can see two things that will help to control the size of $|T(u) - T(v)|$: first, restrict to a small enough region around x_0 so that f does not vary much on that region, and second, make the interval in which t varies small. We address the first problem below by considering the derivative of f.