Math 322 lecture for Wednesday, Week 5

NONHOMOGENEOUS SYSTEMS

Proposition. Let $A \in M_n(F)$ and consider the system

$$x'(t) = Ax(t) + b(t)$$

where $t \mapsto b(t) \in F^n$ is continuous. The solution with initial condition x_0 is

$$x(t) = e^{At}x_0 + e^{At}\int_{s=0}^{t} e^{-As}b(s)\,ds.$$

The solution is unique.

Proof. Given in the last lecture: just take the derivative of the above expression. Uniqueness is a homework problem. □

Note. Our text has references for a system as in the Proposition but for which $A = A(t)$, i.e., A varies with t, too.

Example. Here is an example from our text for an equation modeling a forced harmonic oscillator:

$$x'' = -x + f(t).$$

Writing $x_1 = x$ and $x_2 = x_1'$, we have

$$x_2' = x_1'' = -x + f(t) = -x_1 + f(t).$$

Hence, we consider the system

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -x_1 + f(t) \end{aligned}$$

or let

$$y := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x \\ x' \end{pmatrix}$$

and consider the system

$$y' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} y + \begin{pmatrix} 0 \\ f(t) \end{pmatrix}.$$

So we apply the proposition with

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad b(t) = \begin{pmatrix} 0 \\ f(t) \end{pmatrix}.$$

Thus,

$$e^{At} = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix},$$

and

$$\begin{aligned} y(t) &= e^{At}y_0 + e^{At}\int_{s=0}^{t} e^{-As}b(s)\,ds \\ &= \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} y_0 \\ &\qquad + \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \int_{s=0}^{t} \begin{pmatrix} \cos(s) & -\sin(s) \\ \sin(s) & \cos(s) \end{pmatrix} \begin{pmatrix} 0 \\ f(s) \end{pmatrix} ds \\ &= \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} y_0 + \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \int_{s=0}^{t} \begin{pmatrix} -f(s)\sin(s) \\ f(s)\cos(s) \end{pmatrix} ds. \end{aligned}$$

The initial condition is $y_0 = (x_1(0), x_2(0)) = (x(0), x'(0))$. We take the first component of the above 2×1 matrix to get the solution:

$$\begin{aligned} x(t) &= x(0)\cos(t) + x'(0)\sin(t) \\ &\qquad + \cos(t)\left(-\int_{s=0}^{t} f(s)\sin(s)\,ds\right) + \sin(t)\left(\int_{s=0}^{t} f(s)\cos(s)\,ds\right) \\ &= x(0)\cos(t) + x'(0)\sin(t) + \int_{s=0}^{t} f(s)\left(-\cos(t)\sin(s) + \sin(t)\cos(s)\right)\,ds. \end{aligned}$$

Now use the sum formula

$$\sin(\theta + \psi) = \cos(\theta)\sin(\psi) + \cos(\psi)\sin(\theta)$$

with $\theta = t$ and $\psi = -s$ to get

$$x(t) = x(0)\cos(t) + x'(0)\sin(t) + \int_{s=0}^{t} f(s)\sin(t - s)\,ds.$$

For a special case, suppose that $f(s) = \cos(\omega t)$. The solution is then

$$x(t) = x(0)\cos(t) + x'(0)\sin(t) + \int_{s=0}^{t} \cos(\omega s)\sin(t - s)\,ds.$$

To integrate this, note that

$$\begin{aligned}\sin(\theta+\psi)+\sin(\theta-\psi) &= \cos(\theta)\sin(\psi)+\cos(\psi)\sin(\theta)\\ &\qquad -\cos(\theta)\sin(\psi)+\cos(\psi)\sin(\theta)\\ &= 2\cos(\psi)\sin(\theta).\end{aligned}$$

Therefore,

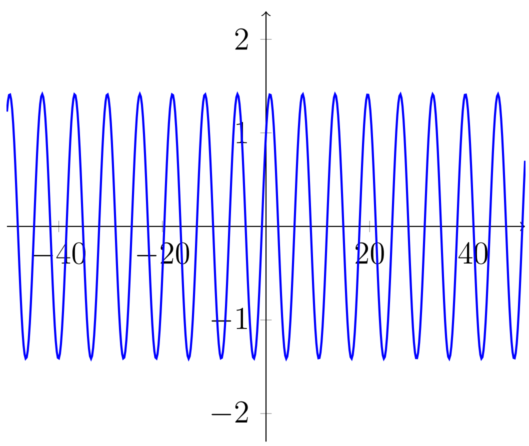
$$\cos(\psi)\sin(\theta) = \frac{1}{2}\left(\sin(\theta+\psi)+\sin(\theta-\psi)\right).$$

It follows that

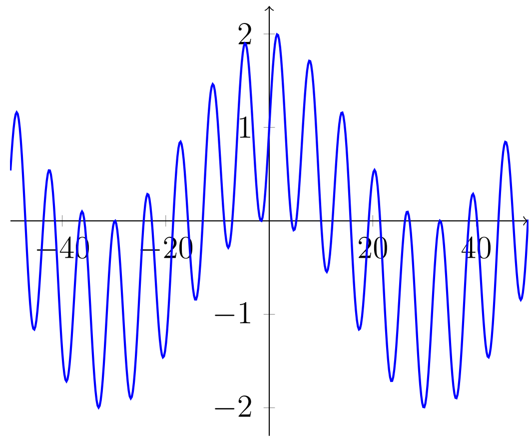
$$\begin{aligned}\int_{s=0}^{t}\cos(\omega s)\sin(t-s)\,ds &= \frac{1}{2}\int_{s=0}^{t}\sin(t+(\omega-1)s)+\sin(t-(\omega+1)s)\,ds\\ &= \frac{1}{2}\left(-\frac{\cos(t+(\omega-1)s)}{\omega-1}+\frac{\cos(t-(\omega+1)s)}{\omega+1}\right)\Bigg|_{s=0}^{t}\\ &= \frac{\cos(\omega t)-\cos(t)}{1-\omega^2}.\end{aligned}$$

So the solution is

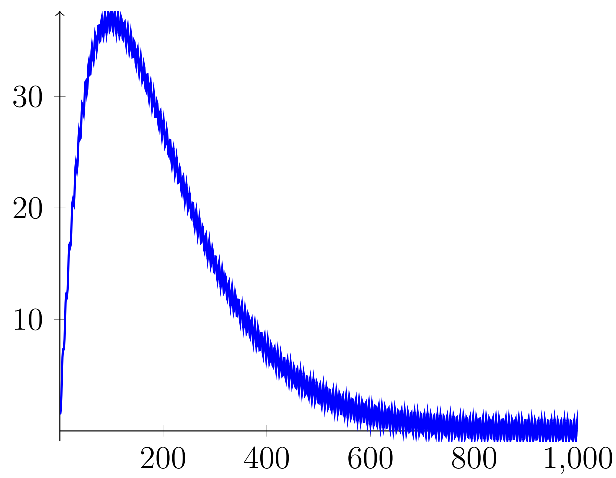
$$x(t) = x(0)\cos(t)+x'(0)\sin(t)+\frac{\cos(\omega t)-\cos(t)}{1-\omega^2}.$$



Unforced: $x(0) = x'(0) = 1,\ f(t) = 0$



$x(0) = x'(0) = 1,\ \omega = 0.1$



$x(0) = x'(0) = 1,\ f(t) = te^{-0.01t}$