Math 322 worksheet for Friday, Week 5

Consider the 4th order linear homogeneous equation with constant coefficients:

$$y^{(iv)} + a_3 y''' + a_2 y'' + a_1 y' + a_0 y = 0 \tag{1}$$

with initial condition $y^{(i)}(0) = b_i$ for $i = 0, 1, 2, 3$.
Let $x_i = y^{(i-1)}$ for $i = 1, 2, 3, 4$, and let $x(t) = (x_1(t), \ldots, x_4(t))$. Use equation (1) to create a linear system:

$$x' = Ax$$

with initial condition $x(0)$.

PROBLEM 1. What is the 4×4 matrix A? And what is $x(0)$ in terms of y?

PROBLEM 2. The solution to the above system is $x(t) = e^{At} x_0$. Suppose you have calculated e^{At}. How do read off the solution to our original equation (1)?

According to the recipe we learned during the first couple of weeks of class, to solve equation (1), we first consider its characteristic polynomial $P(x) = x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0$. We would like to compare $P(x)$ to $p_A(x) := \det(A - xI_4)$, the characteristic polynomial for the matrix A.

PROBLEM 3. Compute $\det(A - xI_4)$ by first performing the following column operations (which don't affect the value of the determinant): add x times the second column to the first column, then add x^2 times the third column to the first column, then add x^3 times the fourth column to the first column. (i) What is the result? The first column should consist of zeros except for the last entry. (ii) What is this last entry? (iii) Compute the determinant by expanding along the first column. What do you get? (iv) What would you get if instead of starting with a 4-th degree equation, we started with an n-th degree equation?

Let λ be an eigenvalue for A, and consider the corresponding eigenspace,

$$E_\lambda = \left\{ v \in F^4 : Av = \lambda v \right\}.$$

PROBLEM 4. Prove that

$$E_\lambda = \operatorname{Span}\left\{ (1, \lambda, \lambda^2, \lambda^3) \right\}.$$

Thus, $\dim E_\lambda = 1$, i.e., the geometric multiplicity of λ is 1. (Hint: let $v = (v_1, \ldots, v_n)$ and then compare components on both sides of the equation $Av = \lambda v$.)

Suppose that A has k distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ over $\mathbb{C}$ with multiplicities $m_1, \ldots, m_k$, respectively. So the characteristic polynomial factors as $p_A(x) = \prod_{i=1}^{k}(\lambda_i - x)^{m_i}$.

PROBLEM 5. Why do we know that the Jordan form for A over the complex numbers is

$$\begin{pmatrix} J_{m_1}(\lambda_1) & & & \\ & J_{m_2}(\lambda_2) & & \text{\Large 0} \\ & \text{\Large 0} & \ddots & \\ & & & J_{m_k}(\lambda_k) \end{pmatrix}?$$

PROBLEM 6. Define the *basic functions* for equation (1) to be

$$\left\{t^j e^{\lambda_i t} : 0 \le j < m_i, 1 \le i \le k\right\}.$$

Prove that every solution to equation (1) is a linear combination of these basic functions.

We would like to show that each of the basic functions is a solution to equation (1). Consider the differential operator $D := \frac{d}{dt}$. We can write equation (1) as

$$P(D)y = 0$$

where $P(D) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$. Further, we know (why?) that

$$P(D) = \prod_{i=1}^{k}(D - \lambda_i)^{m_i}.$$

Problem 7.

(a) Prove by induction that for every sufficiently differentiable function $f(t)$, we have

$$(D - \lambda)^k(f(t)e^{\lambda t}) = e^{\lambda t}D^k f(t)$$

for $k \geq 0$.

(b) Show that it follows that

$$P(D)(f(t)e^{\lambda t}) = e^{\lambda t} P(D + \lambda) f(t).$$

(c) Use these results to show that each basic function is a solution to equation (1).

Finally, we'd like to show that each solution to equation (1) with the given initial condition is a *unique* linear combination of the basic functions. To do so, list the basic functions in some order f_1, f_2, f_3, f_4. For each $\alpha = (\alpha_1, \alpha_{,}\alpha_3, \alpha_4) \in \mathbb{C}^4$, consider the solution

$$s_\alpha(t) = \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 + \alpha_4 f_4,$$

and, in general, define

$$\begin{aligned} \phi_\alpha \colon \mathbb{C}^4 &\to \mathbb{C}^4 \\ \alpha &\mapsto (s(0), s'(0), s''(0), s'''(0)). \end{aligned}$$

It's clear that ϕ is linear (since differentiation and evaluation are linear).

Problem 8. You have already shown that is ϕ surjective. How? Why does it then follow that ϕ injective? How does this prove uniqueness?