Math 322 lecture for Wednesday, Week 4

Review of diagonalization. For a diagonal matrix $D = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$, we have

$$De_i = \lambda_i e_i$$

for each standard basis vector e_i. If $A \in M_n(F)$ is not diagonal, we look for linearly independent vectors that behave like the e_i above:

$$Av_i = \lambda_i v_i.$$

If we can find n of these vectors, then changing to the basis $\{v_1, \dots, v_n\}$, these v_i are transformed to the standard basis vectors in the new coordinates, and A is diagonalized.

Therefore, we look for vectors $v \neq 0$ such that

$$Av = \lambda v$$

for some $\lambda \in F$. We have

$$Av = \lambda v \quad \Leftrightarrow \quad (A - \lambda I_n)v = 0 \quad \Leftrightarrow \quad v \in \ker(A - \lambda I_n).$$

The kernel is nonzero if and only if $\det(A - \lambda I_n) = 0$. So to find suitable λ, the *eigenvalues*, we consider the *characteristic polynomial*

$$p(x) = \det(A - xI_n) = \prod_{j=1}^{n} (\lambda_j - x) = \prod_{j=1}^{\ell} (\mu_j - x)^{k_j}.$$

In the expression on the far right, repeated eigenvalues are grouped together (so each μ_j is equal to some λ_t). The *algebraic multiplicity* of the eigenvalue μ_j is k_j. The eigenvectors corresponding to μ_j form a subspace of F^n called the *eigenspace* for μ_j:

$$E_{\mu_j} := \ker(A - \mu_j I_n).$$

The dimension of E_{μ_j} is the *geometric multiplicity* of μ_j. We always have that the geometric multiplicity is at most the algebraic multiplicity:

$$\dim E_{\mu_j} \leq k_j.$$

The matrix A is diagonalizable if and only if there is a basis consisting of eigenvectors, and that happens exactly when the geometric multiplicity of each eigenvalue equals it algebraic multiplicity. If that is not the case, we can still choose bases for each eigenspace, but we are then left with the task of completing this set to a full basis for F^n. By choosing correctly, we can assure that A has a nice form.

JORDAN FORM

Let $\lambda \in F$. A $k \times k$ *Jordan block* for λ is a $k \times k$ matrix with λ appearing along the diagonal and 1s appearing on the superdiagonal:

$$J_k(\lambda) := \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \Large 0 \\ & & \lambda & 1 & \\ & \Large 0 & & \ddots & 1 \\ & & & & \lambda \end{pmatrix}.$$

For example,

$$J_4(2) = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

A *Jordan matrix* is a square block-diagonal matrix with Jordan matrices along the diagonal:

$$J := \begin{pmatrix} J_{k_1}(\lambda_1) & & & & \\ & J_{k_2}(\lambda_2) & & & \Large 0 \\ & & J_{k_3}(\lambda_3) & & \\ & \Large 0 & & \ddots & \\ & & & & J_{k_\ell}(\lambda_\ell). \end{pmatrix}.$$

For example, the following is a Jordan matrix:

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2+3i \end{pmatrix}$$

with Jordan blocks $J_1(2), J_1(2), J_3(4), J_2(i)$ and $J_1(2+3i)$.

A diagonal matrix is a Jordan matrix whose Jordan blocks are all 1×1.

Theorem. Let $A \in M_n(\mathbb{C})$. Then there exists an invertible matrix $P \in M_n(\mathbb{C})$ such that $P^{-1}AP = J$ where J is a Jordan matrix. The matrix J is called the *Jordan form for A*. It is unique up to a permutation of the Jordan blocks. The diagonal entries of J are exactly the eigenvalues of A repeated according to their algebraic multiplicities (the number of times the eigenvalue appears in a factorization of the characteristic polynomial of A over $\mathbb{C}$). The number of blocks having a particular eigenvalue λ along the diagonal is the geometric multiplicity of λ (i.e., $\dim(A - \lambda I_n)$).

Example. A matrix is diagonalizable if and only if each of its Jordan blocks is 1×1. For example, we know

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

is not diagonalizable since it is already in Jordan form and it's not diagonal. The matrix A has one eigenvalue, 1, of multiplicity 2, but the dimension of the eigenspace for 1 is 1-dimensional:

$$\ker(A - 1 \cdot I_2) = \ker \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \{(x, 0) : x \in F\},$$

which has basis $\{(1, 0)\}$. As claimed the number of Jordan blocks for 1 is the geometric multiplicity of 1.

Jordan form over the reals. Now suppose that $A \in M_n(\mathbb{R})$. Then it turns out that we can conjugate A via a real matrix it to a real matrix that is almost as nice as the Jordan form over $\mathbb{C}$. Since A is defined over the reals, its nonreal eigenvalues appear in conjugate pairs, and it turns out that each $k \times k$ Jordan block for $\lambda = a + bi$ has a corresponding $k \times k$ Jordan block for $\bar{\lambda} = a - bi$ of the same dimension. We can combine these blocks and change basis to get a corresponding $2k \times 2k$ block matrix with 2×2 blocks of the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

along the diagonal, and the 2×2 identity matrix I_2 appearing along the super diagonal.

For instance, the Jordan matrix

$$\begin{pmatrix} \lambda & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{\lambda} & 1 & 0 \\ 0 & 0 & 0 & 0 & \bar{\lambda} & 1 \\ 0 & 0 & 0 & 0 & 0 & \bar{\lambda} \end{pmatrix}$$

where $\lambda = a + bi$ can be conjugated to the form

$$\begin{pmatrix} a & -b & 1 & 0 & 0 & 0 \\ b & a & 0 & 1 & 0 & 0 \\ 0 & 0 & a & -b & 1 & 0 \\ 0 & 0 & b & a & 0 & 1 \\ 0 & 0 & 0 & 0 & a & -b \\ 0 & 0 & 0 & 0 & b & a \end{pmatrix}$$

If $A \in M_n(\mathbb{R})$, there exists an invertible $P \in M_n(\mathbb{R})$ such that $P^{-1}AP = J$ where J consists of Jordan blocks—the usual ones for real eigenvalues, and these modified block matrices for conjugate pairs of complex eigenvalues. The form is unique up to permutation of the blocks and swaps

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \quad \longleftrightarrow \quad \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

We will call it the *real Jordan form* for A. Here is a typical real Jordan form for a real matrix:

$$\begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 3 \end{pmatrix}$$

There are two Jordan blocks for 4: one is 1×1 and one is 3×3. The other eigenvalues for this matrix are $3 + 2i$ and $3 - 2i$, each of which appears with multiplicity 3. Notice there are two real Jordan blocks for the pair $3 \pm 2i$, one is 2×2 and the other is 4×4.