Math 322 lecture for Monday, Week 4

LINEAR SYSTEMS IN $\mathbb{R}^2$

Let $A \in M_2(\mathbb{R})$. The characteristic polynomial has real coefficients and degree 2. That means that if λ is a complex eigenvalue for A (with nonzero imaginary part), then so is its conjugate $\bar{\lambda}$. Otherwise, A either has two distinct real eigenvalues or one real eigenvalue with multiplicity 2. In order to exponentiate A, it would be nice to conjugate A (i.e., apply the mapping $A \to P^{-1}AP$ for some P) to a matrix that is close to being diagonal. We will discuss the Jordan form more carefully later, but for now it suffices to know that there exists an invertible real matrix P such that $P^{-1}AP$ has one of the three possible forms below:

$$\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}, \qquad \begin{pmatrix} u & 1 \\ 0 & u \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} a & -b \\ b & a \end{pmatrix},$$

where $u, v, a, b \in \mathbb{R}$. The first case occurs when A has eigenvalues u and v (including the case where $u = v$ occurs with multiplicity 2) and A is diagonalizable. The second case occurs when A has the real eigenvalue u with multiplicity 2 but the corresponding eigenspace only has dimension 1. The last case occurs when A has a pair of complex eigenvalues $\lambda = a + bi$ and $\lambda = a - bi$. (If we were working over $\mathbb{C}$, then in this last case A could be conjugated to the diagonal matrix $\operatorname{diag}(\lambda, \bar{\lambda})$, as we will discuss below.)

To solve two-dimensional linear systems, we need to exponentiate matrices with these forms. The first is easy:

$$\exp \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} = \begin{pmatrix} e^u & 0 \\ 0 & e^v \end{pmatrix}.$$

For the second, let's exponentiate a slightly more general matrix:

$$B := \begin{pmatrix} u & v \\ 0 & u \end{pmatrix}.$$

Let

$$C = \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix},$$

and note that (i) $B = uI + C$, (ii) $C^k = 0$ for $k > 1$, and (iii) uI and C commute. It follows that

$$
\begin{aligned}
e^B = e^{uI+C} = e^{uI}e^C &= \begin{pmatrix} e^u & 0 \\ 0 & e^u \end{pmatrix} e^C = e^u I e^C = e^u e^C \\
&= e^u \left(I + C + \frac{1}{2}C^2 + \frac{1}{3!}C^3 + \dots \right) \\
&= e^u \left(I + C \right) \\
&= \begin{pmatrix} e^u & ve^u \\ 0 & e^u \end{pmatrix}.
\end{aligned}
$$

Now consider the last case, in which

$$J = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Letting

$$Q = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$$

we have

$$
\begin{aligned}
Q^{-1}JQ &= \frac{1}{2i} \begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \\
&= \frac{1}{2i} \begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix} \begin{pmatrix} ai - b & -ai - b \\ a + bi & a - bi \end{pmatrix} \\
&= \frac{1}{2i} \begin{pmatrix} 2ai - 2b & 0 \\ 0 & 2ai + 2b \end{pmatrix} \\
&= \begin{pmatrix} a + bi & 0 \\ 0 & a - bi \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}.
\end{aligned}
$$

Therefore, using the fact that

$$e^{\lambda t} = e^{at+bti} = e^{at}(\cos(bt) + i\sin(bt)) \quad \text{and} \quad e^{\bar{\lambda}t} = e^{at-bti} = e^{at}(\cos(bt) - i\sin(bt)),$$

we have

$$e^{\begin{pmatrix} a & -b \\ b & a \end{pmatrix} t} = Qe^{\operatorname{diag}(\lambda,\bar{\lambda})t}Q^{-1}$$

$$= \frac{1}{2i} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\bar{\lambda} t} \end{pmatrix} \begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix}$$

$$= \frac{1}{2i} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{\lambda t} & ie^{\lambda t} \\ -e^{\bar{\lambda} t} & ie^{\bar{\lambda} t} \end{pmatrix}$$

$$= \frac{1}{2i} \begin{pmatrix} ie^{\lambda t} + ie^{\bar{\lambda} t} & -e^{\lambda t} + e^{\bar{\lambda} t} \\ e^{\lambda t} - e^{\bar{\lambda} t} & ie^{\lambda t} + ie^{\bar{\lambda} t} \end{pmatrix}$$

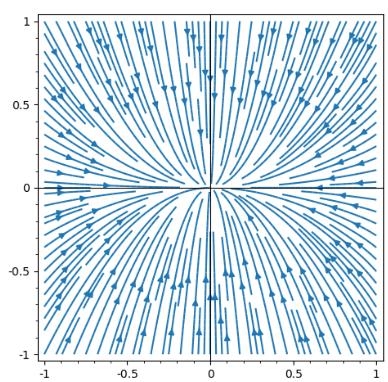
$$= e^{at} \begin{pmatrix} \cos(bt) & -\sin(bt) \\ \sin(bt) & \cos(bt) \end{pmatrix}.$$

Let's look at the corresponding systems of differential equations and their solutions with initial condition x_0:

If $J = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$ then the solution is

$$x(t) = \begin{pmatrix} e^{ut} & 0 \\ 0 & e^{vt} \end{pmatrix} x_0.$$

If both u and v are negative, the origin is a *stable node* ($u = -1, v = -2$ displayed):



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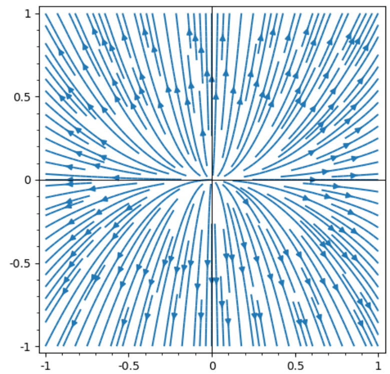
If u and v are both positive, the origin is an *unstable node* ($u = 1, v = 2$ displayed):



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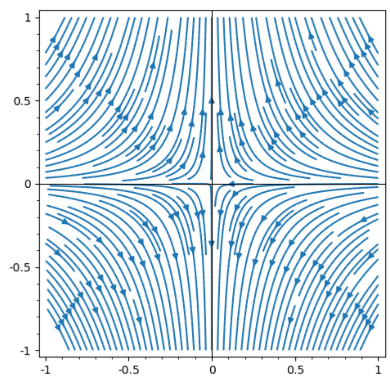
If one of u and v is negative and the other is positive, the origin is a *saddle point* ($u = -1, v = 2$ displayed):



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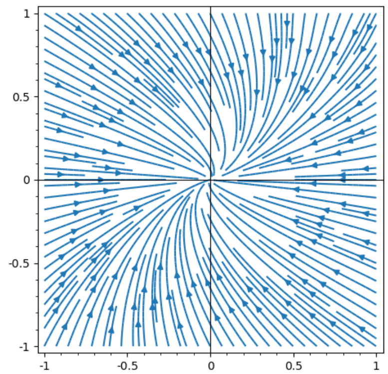
If $J = \begin{pmatrix} u & 1 \\ 0 & u \end{pmatrix}$ then

$$e^{Jt} = \exp \begin{pmatrix} u & t \\ 0 & u \end{pmatrix} = \begin{pmatrix} e^u & te^u \\ 0 & e^u \end{pmatrix}$$

and the solution is

$$x(t) = \begin{pmatrix} e^{ut} & te^{ut} \\ 0 & e^{ut} \end{pmatrix} x_0.$$

If $u < 0$, the origin is a *stable node* ($u = -2$ displayed):



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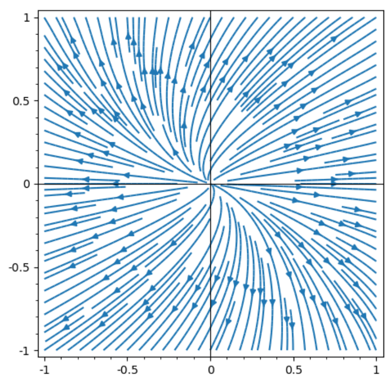
and if it is positive, then the origin is an *unstable node* ($u = 2$ displayed):



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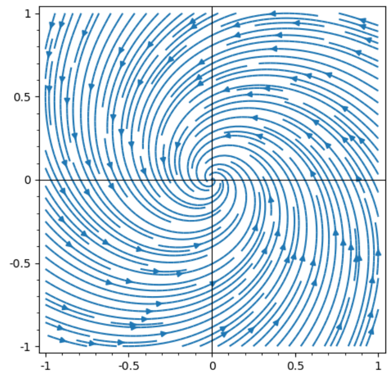
If $J = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ then the solution is

$$x(t) = e^{at} \begin{pmatrix} \cos(b) & -\sin(b) \\ \sin(b) & \cos(b) \end{pmatrix} x_0.$$

If $a < 0$, then each solution spirals into the origin and we say the origin is a *stable focus* ($a = -1, b = 2$ displayed):



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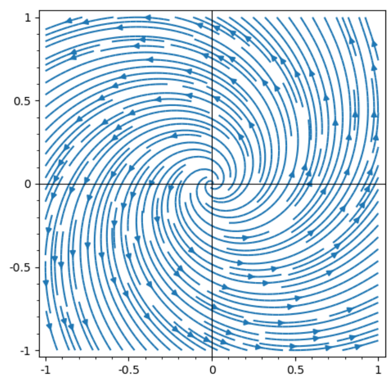
If $a > 0$, then each solution spirals away from the origin, and we say the origin is an *unstable focus* ($a = 1, b = 2$ displayed):



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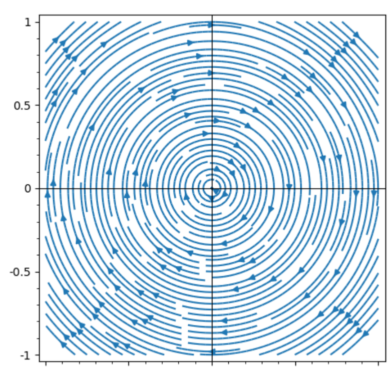
If $a = 0$, each solution goes in a circle about the origin, and we say that the system has a *center* at the origin ($a = 0, b = -2$ displayed):



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In any of these cases, if $b > 0$ the motion is counterclockwise, and if $b < 0$, the motion is clockwise.

We've discussed all cases in which both eigenvalues are nonzero. If either of the eigenvalues is zero, i.e., if $\det(A) = 0$, then the origin is a *degenerate equilibrium point*. See our text for pictures of these systems.

Lemma. Let $A \in M_n(F)$ with eigenvalues $\lambda_1, \ldots, \lambda_n$. Then

(a) $\mathrm{trace}(A) := \sum_{i=1}^n A_{ii} = \sum_{i=1}^n \lambda_i$ and $\det(A) = \prod_{i=1}^n \lambda_i$.

(b) Consider the characteristic polynomial of A:

$$p(x) = \det(A - xI_n).$$

Then the coefficient of x^{n-1} in $p(x)$ is $(-1)^{n-1}\mathrm{trace}(A)$ and the constant term of $p(x)$ is $\det(A)$.[1]

Proof. Recall that for all $C, D \in M_n(F)$, we have

$$\mathrm{trace}(CD) = \mathrm{trace}(DC)$$

and

$$\det(CD) = \det(C)\det(D) = \det(D)\det(C) = \det(DC).$$

Therefore, for all invertible $P \in M_n(F)$,

$$\mathrm{trace}(P^{-1}AP) = \mathrm{trace}(A) \quad \text{and} \quad \det(P^{-1}AP) = \det(A).$$

Further, the characteristic polynomial is not affected by conjugation:

$$\det(P^{-1}AP - xI_n) = \det(P^{-1}(A - xI_n)P) = \det(P^{-1})\det(A - xI_n)\det(P) = \det(A - xI).$$

Therefore, we may assume that A is in Jordan form—an upper triangular matrix. Considering $p(x) = \det(A - xI)$, we see the diagonal entries are the eigenvalues, $\lambda_1 \ldots, \lambda_n$. Part (a) follows. Next, consider the characteristic polynomial

$$\det(A - xI_x) = p(x) = (\lambda_1 - x) \cdots (\lambda_n - x).$$

Expanding the right-hand side, we see that the coefficient of x^{n-1} is $\mathrm{trace}(A)$. Setting $x = 0$ in the above equation then completes the proof of part (b). $\square$

Let's now go back to the case $n = 2$. Let $\tau := \mathrm{trace}(A)$ and $\delta := \det(A)$. Up to conjugation, there are three possibilities:

[1] The characteristic polynomials is sometimes defined to be $p(x) = \det(xI_n - A)$. In that case, the coefficient of x^{n-1} is $-\mathrm{trace}(A)$. The constant term is again $\det(A)$.

$$\begin{array}{ccc}
\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} & \begin{pmatrix} u & 1 \\ 0 & u \end{pmatrix} & \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \\
\tau = u + v & \tau = 2u & \tau = 2a \\
\delta = uv & \delta = u^2 & \delta = a^2 + b^2
\end{array}$$

The characteristic polynomial is

$$p(x) = x^2 - \tau x + \delta.$$

So the eigenvalues are

$$\frac{\tau \pm \sqrt{\tau^2 - 4\delta}}{2}. \tag{1}$$

Theorem. (p. 25)

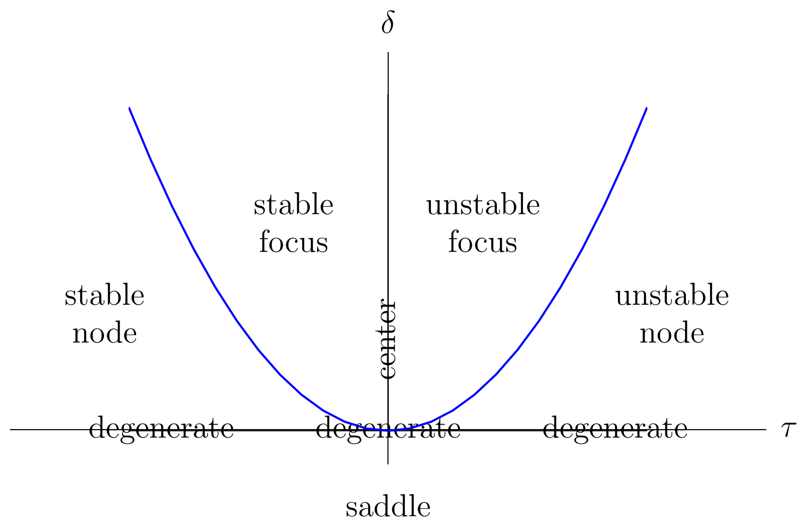
(a) If $\delta < 0$, then the origin is a saddle point.

(b) If $\delta > 0$ and $\tau^2 - 4\delta \geq 0$, then the origin is a stable node if $\tau < 0$ and an unstable node if $\tau > 0$. (Note that in this case, the conditions $\delta > 0$ and $\tau^2 - 4\delta \geq 0$ imply $\tau \neq 0$.)

(c) If $\delta > 0$ and $\tau^2 - 4\delta < 0$, then the origin is a stable focus if $\tau < 0$, an unstable focus if $\tau > 0$, or a center if $\tau = 0$ (in which case $\tau - 4\delta < 0$ is automatic).

Proof. If $\delta < 0$, then equation 1 shows that one eigenvalue is positive and the other is negative. Hence, the origin is a saddle point. That proves the first part. The others follow similarly. $\square$

Calling a stable node or focus a *sink* and calling an unstable node or focus a *source*, we get the following diagram: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