Math 322 lecture for Friday, Week 4

Exponentiation of Jordan matrix

To solve the linear system $x' = Ax$, we need to compute e^{At}. If $P^{-1}AP = J$ where J is the Jordan form of A, then $e^{At} = Pe^{Jt}P^{-1}$. Then, to exponentiate J, we must exponentiate each of its blocks. If

$$J := \begin{pmatrix} J_{k_1}(\lambda_1) & & & & \\ & J_{k_2}(\lambda_2) & & \text{\huge 0} & \\ & & J_{k_3}(\lambda_3) & & \\ & \text{\huge 0} & & \ddots & \\ & & & & J_{k_\ell}(\lambda_\ell) \end{pmatrix},$$

then

$$e^{Jt} := \begin{pmatrix} e^{J_{k_1}(\lambda_1)t} & & & & \\ & e^{J_{k_2}(\lambda_2)t} & & \text{\huge 0} & \\ & & e^{J_{k_3}(\lambda_3)t} & & \\ & \text{\huge 0} & & \ddots & \\ & & & & e^{J_{k_\ell}(\lambda_\ell t)} \end{pmatrix}.$$

Thus, we are reduced to exponentiating Jordan blocks, which we talk about here, starting with an example. Let $\lambda \in F$ and consider the Jordan block

$$J_4(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \lambda I_4 + N_4$$

where

$$N_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since λI_4 and N_4 commute,

$$e^{J_4(\lambda)t} = e^{(\lambda I_4 + N_4)t} = e^{\lambda t I_4} e^{tN_4}.$$

As usual,

$$e^{\lambda t I_4} = \begin{pmatrix} e^{\lambda t} & 0 & 0 & 0 \\ 0 & e^{\lambda t} & 0 & 0 \\ 0 & 0 & e^{\lambda t} & 0 \\ 0 & 0 & 0 & e^{\lambda t} \end{pmatrix} = e^{\lambda t} I_4.$$

So we are left with computing e^{tN_4}:

$$e^{N_4 t} = I_4 + tN_4 + \frac{t^2}{2!}N_4^2 + \frac{t^3}{3!}N_4^3 + \frac{t^4}{4!}N_4^4 + \frac{t^5}{5!}N_4^5 + \cdots$$

Consider the powers of N_4:

$$N_4^2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$N_4^3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$N_4^4 = 0.$$

All higher powers of N_4 are 0. Notice how as we take powers, the diagonal of 1s climbs up to the right along successively higher diagonals.

Returning to the calculation,

$$\begin{aligned} e^{J_4(\lambda t)} &= e^{\lambda t}\left(I_4 + tN_4 + \frac{t^2}{2!}N_4^2 + \frac{t^3}{3!}N_4^3\right) \\ &= e^{\lambda t}\begin{pmatrix} 1 & t & \frac{t^2}{2!} & \frac{t^3}{3!} \\ 0 & 1 & t & \frac{t^2}{2!} \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2!}e^{\lambda t} & \frac{t^3}{3!}e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2!}e^{\lambda t} \\ 0 & 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & 0 & e^{\lambda t} \end{pmatrix}. \end{aligned}$$

For instance, the solution to $x' = J_4(\lambda)x$ with initial condition $x_0 = (4, 3, 2, 1)$ is

$$
\begin{aligned}
x(t) &= e^{J_4(\lambda)t}x_0 \\
&= e^{\lambda t}\begin{pmatrix} 1 & t & \frac{t^2}{2!} & \frac{t^3}{3!} \\ 0 & 1 & t & \frac{t^2}{2!} \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix} \\
&= e^{\lambda t}\begin{pmatrix} 4 + 3t + 2\frac{t^2}{2!} + \frac{t^3}{3!} \\ 3 + 2t + \frac{t^2}{3!} \\ 2 + t \\ 1 \end{pmatrix},
\end{aligned}
$$

or

$$
x(t) = e^{\lambda t}\left(4 + 3t + 2\frac{t^2}{2!} + \frac{t^3}{3!},\ 3 + 2t + \frac{t^2}{3!},\ 2 + t,\ 1\right).
$$

Now consider a general Jordan block:

$$
J_k(\lambda) = \lambda I_k + N_k
$$

where N_k is the matrix with 1s along the superdiagonal. As before, taking powers of N_k causes the diagonal of 1 to march up to the right, and we get $N_k^k = 0$. A matrix N such that $N^k = 0$ is called *nilpotent*. The minimum k such that $N^k = 0$ is the *degree* of nilpotency. Thus, N_k is nilpotent of degree k. We have

$$
\begin{aligned}
e^{J_k(\lambda)t} &= e^{(\lambda I_k + N_k)t} = e^{\lambda t I_k}e^{N_k t} \\
&= e^{\lambda t}\left(I_k + tN_k + \frac{t^2}{2}N_k^2 + \frac{t^3}{3!}N_k^3 + \cdots + \frac{t^{k-1}}{(k-1)!}N_k^{k-1}\right) \\
&= e^{\lambda t}\begin{pmatrix} 1 & t & \frac{t^2}{2!} & \dots & \dots & \frac{t^{k-1}}{(k-1)!} \\ 0 & 1 & t & \dots & \dots & \frac{t^{k-2}}{(k-2)!} \\ 0 & 0 & 1 & \dots & \dots & \frac{t^{k-3}}{(k-3)!} \\ & \ddots & & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & 0 & 1 & t \\ 0 & \dots & \dots & \dots & 0 & 1 \end{pmatrix}.
\end{aligned}
$$

Note. If the real part of λ is negative, notice how

$$\lim_{t\to\infty} e^{J_k(\lambda)t} = 0.$$

Working exclusively over the reals, we will need to exponentiate Jordan blocks corresponding to pairs of conjugate eigenvalues. Let

$$M := \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

and consider a real Jordan block for $\lambda = a + bi$ with $b \neq 0$:

$$J := \begin{pmatrix} M & I_2 & 0 & \dots & \dots & 0 \\ 0 & M & I_2 & \dots & \dots & 0 \\ 0 & 0 & M & \dots & \dots & 0 \\ & \ddots & & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & 0 & M & I_2 \\ 0 & \dots & \dots & \dots & 0 & M \end{pmatrix}.$$

To exponentiate, let

$$R := \begin{pmatrix} \cos(bt) & -\sin(bt) \\ \sin(bt) & \cos(bt) \end{pmatrix}.$$

So

$$e^{Mt} = e^{at}R.$$

By an argument that is essentially the same as just given above, we get the matrix of 2×2 blocks

$$e^{Jt} = e^{at} \begin{pmatrix} R & tR & \frac{t^2}{2!}R & \dots & \dots & \frac{t^{k-1}}{(k-1)!}R \\ 0 & R & tR & \dots & \dots & \frac{t^{k-2}}{(k-2)!}R \\ 0 & 0 & R & \dots & \dots & \frac{t^{k-3}}{(k-3)!}R \\ & \ddots & & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & 0 & R & tR \\ 0 & \dots & \dots & \dots & 0 & R \end{pmatrix}.$$

Again, notice that if $\mathrm{Re}(\lambda) = a < 0$, then

$$\lim_{t\to\infty} e^{Jt} = 0.$$

Algorithm for computing the Jordan form. Our book has a careful discussion of an algorithm for computing the Jordan form of a matrix A. We will not go into the details (unless there is demand for it!). Here, we'll give over a couple of points, though. To start the algorithm, compute the eigenvalues of the matrix by finding the zeros of the characteristic polynomial. We would like to know the number of Jordan blocks for each eigenvalue and their sizes. The key to this is as follows: Let λ be an eigenvalue, and consider the sequence of integers

$$\delta_\ell := \delta_\ell(\lambda) := \dim\ker(A - \lambda I)^\ell$$

for $\ell = 0, 1, 2, \dots$. **These δ_ℓ are invariant with respect to conjugation, so we might as well imagine that A is in Jordan form already and work block-by-block.** For a Jordan block $J_k(\mu)$ with $\mu \neq \lambda$,

$$\ker(J_k(\mu) - \lambda I)^\ell = 0$$

for all ℓ since each diagonal entry of each power is nonzero. So the $\delta_\ell(\lambda)$ for any block like this are all 0. Now consider each Jordan block of the form $J_k(\lambda)$. We have

$$\ker(J_k(\lambda) - \lambda I)^\ell = \ker N_k^\ell$$

where N_k is the nilpotent matrix from earlier. Thinking about the form of N_k^ℓ is it easy to see that the δ_ℓ sequence for blocks like these is

$$\delta_\ell = \begin{cases} \ell & \text{for } 0 \le \ell \le k, \\ k & \text{for } \ell > k. \end{cases}$$

See Figure 1 for the case where $k = 4$.

The $\delta_\ell(\lambda)$-sequence for A is the sum of the $\delta_\ell(\lambda)$-sequences for each of its Jordan blocks. For instance, $\delta_1(\lambda)$ for A is the number of its Jordan blocks for λ—we've just seen that each of these contributes its $\delta_1 = 1$ to the count. With just a little more thought (see our text), letting ν_k be the number of $k \times k$ Jordan blocks for λ for the $n \times n$ matrix A, we get

$$\nu_k = \begin{cases} 2\delta_1 - \delta_2 & \text{for } k = 1, \\ 2\delta_k - \delta_{k+1} - \delta_{k-1} & \text{for } 1 < k < n, \\ \delta_n - \delta_{n-1} & \text{for } k = n. \end{cases}$$

The point is that the numbers of Jordan blocks of each size for each eigenvalue are determined by the δ-sequences, i.e, by the sequence of dimensions of the kernels, $\ker(A - \lambda I)^\ell$.

k	$(A-\lambda I)^k$	**basis for kernel**	**dimension**
1	$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	e_1	1
2	$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	e_1, e_2	2
3	$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	e_1, e_2, e_3	3
4	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	e_1, e_2, e_3, e_4	4.

Figure 1: The case where $A = J_4(\lambda)$.

To actually conjugate A to Jordan form, for each eigenvalue λ, we consider the tower of subspaces

$$\ker(A-\lambda I) \subseteq \ker(A-\lambda I)^2 \subseteq \ker(A-\lambda I)^3 \subseteq \dots$$

Starting at the leftmost kernel in this tower of subsets, we could successively build bases for these kernels, adding vectors as we move to the right, as we could see earlier in the case where $A = J_4(\lambda)$. Appropriately chosen, these vectors are called *generalized eigenvectors*. We use them as columns of a matrix P so that $P^{-1}AP$ is the Jordan form for A

Let's consider the case where

$$A = J_4(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}.$$

Notice that we have

$$Ae_1 = \lambda e_1,\ Ae_2 = e_1 + \lambda e_2,\ Ae_3 = e_2 + \lambda e_3,\ Ae_4 = e_3 + \lambda e_4.$$

Therefore,

$$\begin{aligned}(A - \lambda I)e_1 &= 0\\(A - \lambda I)e_2 &= e_1\\(A - \lambda I)e_3 &= e_2\\(A - \lambda I)e_4 &= e_3,\end{aligned}$$

and $(A-\lambda I)^{i+1}e_i = 0$ for $i = 2, 3, 4$. So if A is not in Jordan form already, we will look for vectors $v_1, \ldots, v_4$ that behave like the e_i, above. We need to solve $(A-\lambda I)v_i = v_{i-1}$ starting with v_1 an eigenvector with eigenvalue λ. These v_i will be columns in the matrix P.