Math 322 lecture for Wednesday, Week 3

From now on, page references are to our text. Recall that we will always be working over the field $F = \mathbb{R}$ or $\mathbb{C}$.

Definition. A sequence (v_k) in a normed vector space $(V, \| \ \|)$ is a *Cauchy sequence* if for all $\varepsilon > 0$ there exists $N \in \mathbb{R}$ such that for all $m, n > N$, we have

$$\|v_n - v_m\| < \varepsilon.$$

A theorem from analysis says that if V is finite-dimensional then it is *complete*: a sequence (v_k) converges if and only if it is a Cauchy sequence.

Lemma. (Weierstrass M-test) Let V and W be normed vector spaces with V finite-dimensional. For each $k \geq 0$, let $f_k \colon W \to V$ be a function. Let $C \subseteq W$, and suppose there exists a sequence $(M_k)_k$ of positive numbers such that

$$\|f_k(x)\| \leq M_k$$

for all $x \in C$ and for all k. Suppose further that $\sum_k M_k$ converges. Then $\sum_k f_k$ is absolutely and uniformly convergent on C.

Proof. A sequence in a normed space over F converges if and only if it's a Cauchy sequence. Let $\varepsilon > 0$. Since $\sum_k M_k$ converges, there exists $N \in \mathbb{R}$ such that for all $n \geq m > N$, we have

$$|\textstyle\sum_{k=0}^{n} M_k - \sum_{k=0}^{m} M_k| = |\sum_{k=m+1}^{n} M_k| < \varepsilon.$$

But then for $n \geq m > N$ is follows that for all $x \in C$

$$\|\textstyle\sum_{k=m+1}^{n} f_k(x)\| \leq \sum_{k=m+1}^{n} \|f_k(x)\| \leq \sum_{k=m+1}^{n} M_k < \varepsilon.$$

Thus $\sum_k f_k$ is uniformly Cauchy. $\square$

We are now ready to prove that it makes sense to exponentiate a matrix:

Theorem. For all $A \in M_n(F)$ and $t_0 > 0$, the function $\mathbb{R} \to M_n(F)$ given by

$$t \mapsto \sum_{k \geq 0} \frac{A^k t^k}{k!}$$

converges absolutely and uniformly for $t \in [-t_0, t_0]$.

Proof. Let $a := \|A\|$ and suppose that $|t| \leq t_0$. Then from Lemma 1 in the previous lecture,

$$\left\|\frac{A^k t^k}{k!}\right\| \leq \frac{\|A\|^k |t|^k}{k!} \leq \frac{\|A\|^k t_0^k}{k!} = \frac{a^k t_0^k}{k!} =: M_k.$$

It follows that

$$\sum_{k \geq 0} M_k = e^{at_0},$$

the usual exponential function. The result follows by the Weierstrass M-test. $\square$

Definition. Let $A \in M_n(F)$ and $t \in \mathbb{R}$. Then

$$e^{At} := \sum_{k \geq 0} \frac{A^k t^k}{k!}.$$

Note: The proof of the previous theorem shows that e^{At} is absolutely convergent and uniformly convergent on any closed interval for t. Further,

$$\|e^{At}\| \leq e^{\|A\||t|}.$$

To rigorously prove this last statement, note that

$$\left\|\sum_{k=0}^{n} \frac{A^k t^k}{k!}\right\| \leq \sum_{k=0}^{n} \left\|\frac{A^k t^k}{k!}\right\| = \sum_{k=0}^{n} \frac{\|A\|^k |t|^k}{k!}$$

The norm is a continuous function and hence commutes with limits, and limits preserve inequalities. It therefore follows that

$$\|e^{At}\| = \left\|\lim_{n \to \infty} \sum_{k=0}^{n} \frac{A^k t^k}{k!}\right\| = \lim_{n \to \infty} \left\|\sum_{k=0}^{n} \frac{A^k t^k}{k!}\right\| = \lim_{n \to \infty} \sum_{k=0}^{n} \frac{\|A\|^k |t|^k}{k!} = e^{\|A\||t||}.$$

Proposition. (p. 13) Let $A, P \in M_n(F)$ with P invertible. Then

$$e^{P^{-1}AP} = P^{-1} e^A P.$$

Proof. Recall the trick from linear algebra:

$$\begin{aligned}(P^{-1}AP)^k &= (P^{-1}AP)(P^{-1}AP)(P^{-1}AP)\cdots(P^{-1}AP)\\ &= P^{-1}A(PP^{-1})A(PP^{-1})A(P\cdots P^{-1})AP\\ &= P^{-1}A^kP.\end{aligned}$$

Therefore,

$$\begin{aligned}e^{P^{-1}AP} &:= \sum_{k\geq 0}\frac{(P^{-1}AP)^k}{k!}\\ &= \sum_{k\geq 0}\left(P^{-1}\frac{A^k}{k!}P\right)\\ &= P^{-1}\left(\sum_{k\geq 0}\frac{A^k}{k!}\right)P\\ &= P^{-1}e^AP.\end{aligned}$$

The matrices P^{-1} and P can be pulled out of the sum since multiplication by these represent linear transformations, which are continuous, and the sum is a limit—limits commute with continuous functions (by definition of continuity). □

Proposition. (p. 13) Let $A, B \in M_n(F)$. If A and B commute, then $e^{(A+B)} = e^Ae^B$.

Proof.

$$\begin{aligned}e^{(A+B)} &= \sum_{n\geq 0}\frac{1}{n!}(A+B)^n\\ &= \sum_{n\geq 0}\frac{1}{n!}\left(\sum_{i+j=n}\frac{n!}{i!j!}A^iB^j\right)\\ &= \sum_{i\geq 0}\frac{1}{i!}A^i\left(\sum_{j\geq 0}\frac{1}{j!}B^j\right)\\ &= e^Ae^B.\end{aligned}$$

□

Corollary. (p. 13) If $A \in M_n(F)$, then

$$e^{-A} = \left(e^A\right)^{-1}.$$

Proof. Since A and $-A$ commute,

$$I_n = e^0 = e^{(A+(-A))} = e^A e^{-A}.$$

$\square$

Example. The above proposition only holds, in general, if the matrices A and B commute. Consider,

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

It is easy to check that $AB \neq BA$.
Since $A^k = 0$ for $k > 1$,

$$e^A = I + A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

and

$$e^B = \sum_{k\geq 0} \frac{1}{k!} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}^k = \sum_{k\geq 0} \frac{1}{k!} \begin{pmatrix} 1^k & 0 \\ 0 & 2^k \end{pmatrix} = \sum_{k\geq 0} \begin{pmatrix} 1/k! & 0 \\ 0 & 2^k/k! \end{pmatrix} = \begin{pmatrix} e & 0 \\ 0 & e^2 \end{pmatrix}.$$

Thus,

$$e^A e^B = \begin{pmatrix} e & e^2 \\ 0 & e^2 \end{pmatrix}.$$

On the other hand, you can check by induction that

$$(A+B)^k = \begin{pmatrix} 1 & 2^k - 1 \\ 0 & 2^k \end{pmatrix}.$$

Hence,

$$e^{A+B} = \sum_{k\geq 0} \frac{1}{k!} \begin{pmatrix} 1 & 2^k - 1 \\ 0 & 2^k \end{pmatrix} = \begin{pmatrix} e & e^2 - e \\ 0 & e^2 \end{pmatrix} \neq e^A e^B.$$