Math 322 lecture for Wednesday, Week 2

Aside on Bernoulli-type equations. Imagine a moving particle with velocity v and a force $F = F(v)$ acting on the particle against its direction of motion—a frictional force. It is reasonable to assume $F(-v) = -F(v)$. Now suppose that F has a power series expansion

$$F(v) = a_0 + a_1 v + a_2 v^2 + \dots.$$

The fact that $F(-v) = -F(v)$ implies that the even terms vanish:

$$F(v) = a_1 v + a_3 v^3 + a_5 v^5 + \dots$$

As a first approximation, we could take

$$F(v) = a_1 v$$

Since force is proportional to acceleration, i.e., $F(v) = \text{constant} \cdot v'$, we can write this model of friction as

$$v' = \alpha v.$$

The solution is $v = e^{\alpha t}$, and for our purposes, we take $\alpha < 0$. The next best approximation is to use the first two terms of the series:

$$v' = \alpha v + \beta v^3,$$

which is a Bernoulli-type equation. Question: what is the behavior of a particle whose motion is governed by this equation?

LHCC. We now continue our discussion of linear homogeneous constant coefficients equations. These have the form

$$y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0,$$

or, more succinctly,

$$P(D)y = 0$$

where $D = d/dt$ and $P(x) = \sum_{i=0}^{n} a_i x^i$. The trick is to look for solutions of the form $y = e^{rt}$. We have $P(D)e^{rt} = P(r)e^{rt}$. So we have a solution of that form exactly for the zeros of P.

Example. Solve

$$y'' - 4y' + 13y = 0$$

with initial conditions $y(0) = 0$ and $y'(0) = 1$.

SOLUTION: Find the zeroes of the characteristic polynomial (quadratic equation to the rescue!):

$$r^2 - 4r + 13 = 0 \quad \Rightarrow \quad r = 2 \pm 3i.$$

So the general solution is

$$y = Ae^{(2+3i)t} + Be^{(2-3i)t}.$$

We would like to express the solution in terms of real numbers:

$$\begin{aligned} y &= Ae^{(2+3i)t} + Be^{(2-3i)t} \\ &= Ae^{2t}(\cos(3t) + i\sin(3t)) + Be^{2t}(\cos(3t) - i\sin(3t)) \\ &= (A+B)e^{2t}\cos(3t) + (A-B)ie^{2t}\sin(3t)) \\ &= ae^{2t}\cos(3t) + be^{2t}\sin(3t). \end{aligned}$$

The general real solution is

$$y = ae^{2t}\cos(3t) + be^{2t}\sin(3t).$$

Now we handle the initial conditions:

$$0 = y(0) = ae^0\cos(0) + be^0\sin(0) = a.$$

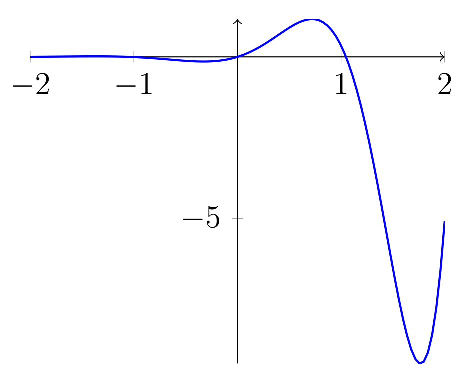
So $y = be^{2t}\sin(3t)$. Then

$$1 = y' = 3be^0\cos(0) + 2be^0\sin(0) = 3b.$$

So the solution is

$$y = \frac{1}{3}e^{2t}\sin(3t)$$

Graph of solution:



There is one final wrinkle in the story: what if $P(r)$ has a repeated root? Say $P(r)$ has a factor of the form $(r-\lambda)^k$. In that case, the general solution will include $a_0 e^{\lambda t} + a_1 t e^{\lambda t} + \cdots + a_k t^{k-1} e^{\lambda t}$. We will be able to understand why this is the case once we move to the higher-dimensional linear theory. For now, you're invited to check that $(D-\lambda)^k t^\ell e^{\lambda t} = 0$ for $0 \leq \ell \leq k-1$ by hand. That way, you'll at least see these are solutions.

Examples.

1. Consider the equation
$$y''' + 6y'' + 12y' + 8y = 0.$$
Its characteristic polynomial is
$$P(r) = r^3 + 6r^2 + 12r + 8 = (r+2)^3.$$
So $P(r)$ as the root $r = -2$ of multiplicity 3. The general solution to the equation is therefore
$$y = ae^{-2t} + bte^{-2t} + ct^2 e^{-2t} = (a + bt + ct^2)e^{-2t}.$$

2. Consider the equation
$$y^{(5)} + 3y^{(4)} + 3y^{(3)} + y^{(2)} = 0.$$
Its characteristic polynomial is
$$P(r) = r^5 + 3r^4 + 3r^3 + r^2 = r^2(r+1)^3.$$
The roots are $r = 0$ with multiplicity 2 and $r = -1$ with multiplicity 3. Notice that the root $r = 0$ will correspond to solutions involving $e^{0 \cdot t} = 1$. The general solution is
$$a_1 + a_2 t + a_3 e^{-t} + a_4 t e^{-t} + a_5 t^2 e^{-t}.$$

3. Say we are considering a LHCC differential equation with characteristic polynomial
$$P(r) = r^3(r-2)^2(r^2+9)^2 = 0.$$
The roots are $r = 0, 2, \pm 3i$ with multiplicities $3, 2, 2$, respectively. The general solution is
$$y = a_1 + a_2 t + a_3 t^2 + b_1 e^{2t} + b_2 t e^{2t} + c_1 \cos(3t) + c_2 \sin(3t) + c_3 t \cos(3t) + c_4 t \sin(3t).$$

V. Method of undetermined coefficients.

We now consider inhomogeneous linear equations with constant coefficients. These have the form

$$P(D)y = f(t).$$

Where P is a polynomial and $D = d/dt$, as before. To solve this equation, we first try to find a particular solution y_p. We then find a general solution y_h to $P(D)y = 0$, the associated homogeneous system. The general solution to the inhomogeneous system is then $y_h + y_p$. The new challenge here is to find the particular solution, y_p. The idea we will use is to guess the form of y_p and adjust parameters. Here is a table that may be of help ("poly" means "polynomial"):

$f(t)$	guess
polynomial	general polynomial of some degree
e^{rt}	ae^{rt}
(poly)e^{rt}	(general poly)e^{rt}
$\cos(\omega t)$ or $\sin(\omega t)$	$a\cos(\omega t) + b\sin(\omega t)$
(poly)$e^{rt}\cos(\omega t)$ or (poly)$e^{rt}\sin(\omega t)$	(gen poly)$e^{rt}\cos(\omega t)$ + (gen poly)$e^{rt}\sin(\omega t)$

Example. Consider the equation

$$y'' - 2y' + y = t^2.$$

We guess a particular equation of the form

$$y = a_0 + a_1 t + a_2 t^2.$$

In that case, we have

$$\begin{aligned} y'' - 2y' + y &= 2a_2 - 2(a_1 + 2a_2 t) + (a_0 + a_1 t + a_2 t^2) \\ &= (2a_2 - 2a_1 + a_0) + (-4a_2 + a_1)t + a_2 t^2. \end{aligned}$$

Set this equal to t^2 and compare coefficients:

$$\begin{aligned} 0 &= 2a_2 - 2a_1 + a_0 \\ 0 &= -4a_2 + a_1 \\ 1 &= a_2. \end{aligned}$$

Solving the system gives

$$a_0 = 6,\ a_1 = 4,\ a_2 = 1.$$

So a particular solution is

$$y_p = 6 + 4t + t^2.$$

(Check!) We now solve the associated homogeneous equation

$$y'' - 2y + y = 0.$$

The characteristic polynomial is

$$r^2 - 2r + 1 = (r-1)^2,$$

which has the zero $r = 1$ with multiplicity 2. So the general solution to the homogeneous system is

$$y_h = ae^t + bte^t.$$

The most general solution to the original equation is then

$$y = y_h + y_p = ae^t + bte^t + 6 + 4t + t^2.$$

Suppose we are given initial conditions $y(0) = 1$ and $y'(0) = -2$. Then

$$\begin{aligned} 1 &= y(0) = a + 6 \\ -2 &= y'(0) = a + b + 4. \end{aligned}$$

Therefore, $a = -5$ and $b = -1$. The solution is

$$y = -5e^t - te^t + 6 + 4t + t^2.$$

Graph of solution: