Math 322 lecture for Friday, Week 1

II. A. Exact equations.

An exact differential equation has the form

$$M(t,y) + N(t,y)\,\frac{dy}{dt} = 0.$$

where

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}.$$

We would like to find a solution that defines y implicitly, i.e., we are looking for a function of the form

$$\Phi(t,y) = 0.$$

If we had such a function, then by the chain rule,

$$0 = \frac{d\Phi}{dt} = \frac{\partial \Phi}{\partial t} + \frac{\partial \Phi}{\partial y}\frac{dy}{dt}.$$

Then Φ would be a solution if

$$M(t,y) = \frac{\partial \Phi}{\partial t} \quad \text{and} \quad N(t,y) = \frac{\partial \Phi}{\partial y}.$$

Note that the conditions on the partials of M and N which are required of an exact equation would then follow necessarily:

$$\frac{\partial M}{\partial y} = \frac{\partial^2 \Phi}{\partial t\,\partial y} = \frac{\partial N}{\partial t}.$$

The trick then is to reverse-engineer this argument. Since $M(t,y) = \dfrac{\partial \Phi}{\partial t}$, we integrate M with respect to t:

$$\Phi(t,y) = \int M(t,y)\,dt =: m(t,y) + f(y)$$

where f is an arbitrary function of y. Then we use the fact that $N(t,y) = \dfrac{\partial \Phi}{\partial y}$ to determine $f(y)$:

$$N(t,y) = \frac{\partial \Phi}{\partial y} = \frac{\partial}{\partial y}(m(t,y) + f(y)).$$

This determines $f(y)$ up to a constant.

Note for those who have seen differential forms: Recall that the differential form ω is *exact* if there is a form ψ such that $d\psi = \omega$. Since $d^2 = 0$, such forms are automatically *closed*: $d\omega = d^2\psi = 0$. In our case, we are considering the 0-form, $\psi = \Phi(t, y)$, and then

$$d\psi = \frac{\partial \Phi}{\partial t}\, dt + \frac{\partial \Phi}{\partial y}\, dy. = M(t, y)\, dt + N(t, y)\, dy.$$

Another way of saying the same thing is that the vector field

$$(t, y) \mapsto (M(t, y), N(t, y))$$

is the gradient vector field $\nabla\Phi$.

Example. Solve

$$\sin(t + y) + (2y + \sin(t + y))y' = 0.$$

The equation is not separable. However, it is exact since

$$\frac{\partial}{\partial y} \sin(t + y) = \cos(t + y) = \frac{\partial}{\partial t}(2y + \sin(t + y)).$$

We have $M(t, y) = \sin(t + y)$ and $N(t, y) = 2y + \sin(t + y)$. To solve the equation, note that

$$\int M(t, y)\, dt = -\cos(t + y) + f(y)$$

for some $f(y)$, and then

$$\frac{\partial}{\partial y}(-\cos(t + y) + f(y)) = N(t, y) = 2y + \sin(t + y)$$

implies that

$$\frac{df}{dy} = 2y.$$

Hence, $f(y) = y^2 + \tilde{c}$. Our final solution is

$$-\cos(t + y) + y^2 = c.$$

Slope fields. Let $y = y(t)$ be the solution to a differential equation $y' = F(y, t)$. The graph of $y(t)$ is a curve. At time t_0, the curve passes through the point $(t_0, y(t_0))$

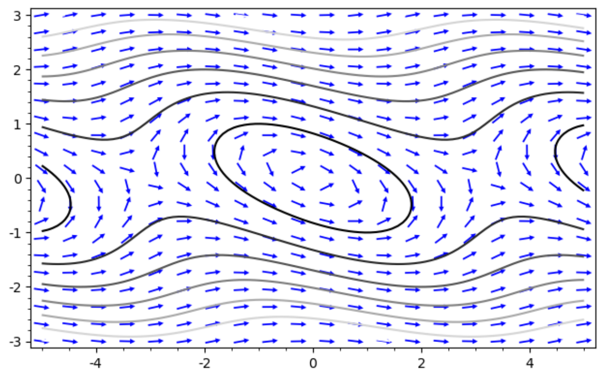

Figure 1: Slope field and solutions for $\sin(t+y) + (2y + \sin(t+y))y' = 0$.

and has slope $y'(t_0) = F(t_0, y(t_0))$. Imagine attaching to each point $(a, b) \in \mathbb{R}^2$ a tiny line segment with slope $F(a, y(a))$. Any solution curve will then be tangent to each line segment it meets. (There will be lots of solutions, depending on the initial condition.) For example, Figure 1 creates the slope field and exhibits several possible solutions. Here is the Sage code used to produce the figure:

```
sage: v = plot_slope_field(-sin(t+y)/(2*y+sin(t+y)),(t,-5,5),(y,-3,3),
...: headaxislength=3, headlength=3,color='blue')
sage: c = contour_plot(-cos(t+y)+y^2,(t,-5,5),(y,-3,3),fill=false)
sage: v + c
Launched png viewer for Graphics object consisting of 2 graphics primitives
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II. B. Exact after multiplying through by integrating factor.

We are again interested in solving

$$M(t,y) + N(t,y)\,\frac{dy}{dt} = 0,$$

but this time, we don't assume that $\partial M/\partial y = \partial N/\partial t$. In that case, we look for a function $\mu(t, y)$ such that

$$\mu(t,y)M(t,y) + \mu(t,y)N(t,y)\,\frac{dy}{dt} = 0,$$

is exact. In fact, μ always exists:

Proof. Let Φ be such that $\Phi(t, y) = 0$ (we can talk about the existence of Φ later, but for now let's assume it exists). Differentiate with respect to t, as before, and use the chain rule

$$0 = \frac{d\Phi}{dt} = \frac{\partial\Phi}{\partial t} + \frac{\partial\Phi}{\partial y}\frac{dy}{dt}.$$

We have

$$\frac{dy}{dt} = -\frac{\partial\Phi/\partial t}{\partial\Phi/\partial y} = -\frac{M(t,y)}{N(t,y)},$$

and, hence,

$$\frac{\partial\Phi/\partial t}{M(t,y)} = \frac{\partial\Phi/\partial y}{N(t,y)} =: \mu(t,y),$$

where we have just now defined μ. It follows that

$$0 = \mu(t,y)M(t,y) + \mu(t,y)N(t,y)\frac{dy}{dt} = \frac{\partial\Phi}{\partial t} + \frac{\partial\Phi}{\partial y}\frac{dy}{dt},$$

which is now exact. $\square$

That's the good news. The bad news is that it might not be easy to find μ. I typical strategy is to assume that μ has a certain form involving parameters, and then try to figure out what values for the parameters will make your equation exact.

Example. Solve

$$ty^2 + 4t^2y + (3t^2y + 4t^3)\frac{dy}{dt} = 0.$$

This equation is not exact. We'll try to find an integrating factor of the form $\mu(t,y) = t^m y^n$. So we would like for

$$(t^m y^n)(ty^2 + 4t^2y) + t^m y^n(3t^2y + 4t^3)\frac{dy}{dt} = 0$$

to be exact. We need

$$\frac{\partial}{\partial y}(t^{m+1}y^{n+2} + 4t^{m+2}y^{n+1}) = \frac{\partial}{\partial t}(3t^{m+2}y^{n+1} + 4t^{m+3}y^n).$$

In other words, we need

$$(n+2)t^{m+1}y^{n+1} + 4(n+1)t^{m+2}y^n = 3(m+2)t^{m+1}y^{n+1} + 4(m+3)t^{m+2}y^n.$$

Equate coefficients:

$$n+2 = 3(m+2) \quad \text{and} \quad 4(n+1) = 4(m+3).$$

Solving this system of linear equations yields $m = -1$ and $n = 1$. Our integrating factor is $\mu(t,y) = y/t$. Ah, ha! That reminds me of the homogeneity trick. In fact, solving for dy/dt in the original equation does give the form $y' = F(y/t)$! So we could have solved this with our earlier machinery. Nevertheless, we'll continue from here. Multiplying through by the integrating factor transforms our original equation into

$$y^3 + 4ty^2 + (3ty^2 + 4t^2 y)\frac{dy}{dt} = 0,$$

which is now exact with

$$M = y^3 + 4ty^2 \quad \text{and} \quad N = 3ty^2 + 4t^2 y.$$

(Check that $\partial M/\partial y = \partial N/\partial t$ to be sure.) Solve the exact equation:

$$\Phi(t, y) = \int M\,dt = ty^3 + 2t^2 y^2 + f(y)$$

implies

$$N(t, y) = \frac{\partial \Phi}{\partial y} = 3ty^2 + 4t^2 y + \frac{df}{dy}.$$

Comparing with $N(t, y)$ shows that $df/dy = 0$. Hence, $f(y) = \tilde{c}$, a constant. Our solution:

$$ty^3 + 2t^2 y^2 = c,$$

(where $c = -\tilde{c}$, is just another constant). Figure 2 give the slope field and several solutions. Figure 3 plots the function $z = ty^3 + 2t^2 y^2$. The level sets of this function are solutions to the differential equation.

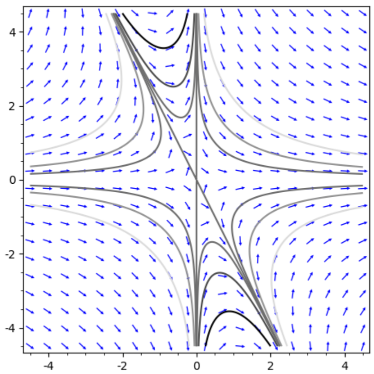

Figure 2: Slope field and solutions to $ty^2 + 4t^2y + (3t^2y + 4t^3)\frac{dy}{dt} = 0$.

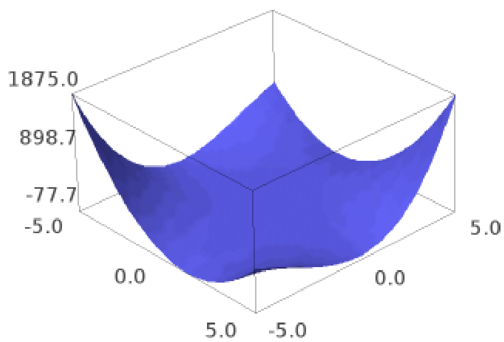

Figure 3: Plot of the surface $z = ty^3 + 2t^2y^2$.