1. Let $F(x, y, z) = (x + 3z, xy^2, y)$ be a vector field, and let $C(t) = (2t, t^3, t + t^2)$ be a parametrized curve with $t \in [0, 1]$. Calculate the flow of F along C in two ways: (i) integrating the flow form for F over C, and (ii) using the classical formula for $\int_C F \cdot dC$.

   SOLUTION: Integrating the flow form:

$$\begin{aligned}
\int_C \omega_F &= \int_C (x + 3z)\,dx + xy^2\,dy + y\,dz \\
&= \int_{[0,1]} (2t + 3(t + t^2))\,d(2t) + (2t)(t^3)^2\,d(t^3) + t^3\,d(t + t^2) \\
&= \int_{[0,1]} (2t + 3(t + t^2))(2)\,dt + (2t)(t^3)^2(3t^2)\,dt + t^3(1 + 2t)\,dt \\
&= \int_0^1 (2t + 3(t + t^2))(2) + (2t)(t^3)^2(3t^2) + t^3(1 + 2t)\,dt \\
&= \int_0^1 6\,t^9 + 2\,t^4 + t^3 + 6\,t^2 + 10\,t\,dt \\
&= \frac{3}{5} + \frac{2}{5} + \frac{1}{4} + 2 + 5 \\
&= \frac{33}{4}.
\end{aligned}$$

   Classical formula:

$$\begin{aligned}
\int_C F \cdot \vec{t} &= \int_0^1 F(C(t)) \cdot C'(t)\,dt \\
&= \int_0^1 F(2t, t^3, t + t^2) \cdot (2, 3t^2, 1 + 2t)\,dt \\
&= \int_0^1 (2t + 3(t + t^2), (2t)(t^3)^2, t^3) \cdot (2, 3t^2, 1 + 2t)\,dt \\
&= \int_0^1 (5t + 3t^2, 2t^7, t^3) \cdot (2, 3t^2, 1 + 2t)\,dt \\
&= \int_0^1 6\,t^9 + 2\,t^4 + t^3 + 6\,t^2 + 10\,t\,dt \\
&= \frac{3}{5} + \frac{2}{5} + \frac{1}{4} + 2 + 5 \\
&= \frac{33}{4}.
\end{aligned}$$

2. Let $F(x,y,z) = (y^2, z, 3x)$ be a vector field, and let $S(u,v) = (u,v,uv)$ be a parametrized curve with $(u,v) \in [0,1]^2$. Calculate the flux of F through S in two ways: (i) integrating the flux form for F over S, and (ii) using the classical formula for $\int_C F \cdot \vec{n}$.

SOLUTION: Integrating the flux form:

$$\begin{aligned}
\int_S \omega^F &= \int_S y^2\, dy \wedge dz - z\, dx \wedge dz + 3x\, dx \wedge dy \\
&= \int_{[0,1]^2} v^2\, dv \wedge d(uv) - uv\, du \wedge d(uv) + 3u\, du \wedge dv \\
&= \int_{[0,1]^2} v^2\, dv \wedge (v\, du + u\, dv) - uv\, du \wedge (v\, du + u\, dv) + 3u\, du \wedge dv \\
&= \int_{[0,1]^2} -v^3\, du \wedge dv - u^2 v\, du \wedge dv + 3u\, du \wedge dv \\
&= \int_0^1 \int_0^1 -v^3 - u^2 v + 3u\, du\, dv \\
&= -\frac{1}{4} - \frac{1}{6} + \frac{3}{2} \\
&= \frac{13}{12}.
\end{aligned}$$

Classical formula:

$$\begin{aligned}
\int_S F \cdot \vec{n} &= \int_0^1 \int_0^1 F(S(u,v)) \cdot (S_u \times S_v) du\, dv \\
&= \int_0^1 \int_0^1 F(u,v,uv) \cdot ((1,0,v) \times (0,1,u))\, du\, dv \\
&= \int_0^1 \int_0^1 (v^2, uv, 3u) \cdot (-v, -u, 1)\, du\, dv \\
&= \int_0^1 \int_0^1 -v^2 - u^2 v + 3u\, du\, dv \\
&= -\frac{1}{4} - \frac{1}{6} + \frac{3}{2} \\
&= \frac{13}{12}.
\end{aligned}$$

3. Thinking of each of the following 1-forms in $\mathbb{R}^3$ as flow forms, find the corresponding vector fields.

   (a) $\omega = x\, dx + \ln(x^2 + z^2)\, dy + (y + xz)\, dz$.

   SOLUTION: $F(x,y,z) = (x, \ln(x^2 + z^2), y + xz)$.

(b) $\eta = \cos(xy)\,dx + \sin(yz)\,dz$.

SOLUTION: $F(x,y,z) = (\cos(xy), 0, \sin(yz))$.

4. Thinking of each of the following 2-forms in $\mathbb{R}^3$ as flux forms, find the corresponding vector fields.

(a) $\eta = -dx \wedge dy + xy\,dx \wedge dz$.

SOLUTION: $F(x,y,z) = (0, -xy, -1)$.

(b) $\omega = dx \wedge (y\,dy - (x+z^2)\,dz)$.

SOLUTION: $F(x,y,z) = (0, x+z^2, y)$.

5. (a) Give an concrete example of a 0-form, η, in $\mathbb{R}^3$, i.e., and element $\eta \in \Omega^0\mathbb{R}^3$.

SOLUTION: A 0-form is just a real-valued function, for example, $\eta = x^2 + y^2 + z^2$.

(b) Interpret integration of $d\eta$ in terms of classical vector calculus. What does Stokes' theorem say in this context?

SOLUTION: The 1-form $d\eta$ is the flow form for the gradient of the function η, i.e.,

$$d\eta = \omega_{\nabla\eta} = 2x\,dx + 2y\,dy + 2z\,dz.$$

Integrating $d\eta$ along a curve with give the flow of the gradient vector field $\nabla\eta$ along the curve. By Stokes' theorem the integral will just give the change in potential, $\eta(C(b)) - \eta(C(a))$, where the domain of C is $[a,b]$.

6. Let ϕ be a function on $\mathbb{R}^3$, and let F be a vector field in $\mathbb{R}^3$. Describe $\operatorname{grad}(\phi)$, $\operatorname{curl}(F)$, and $\operatorname{div}(F)$ using flow forms, flux forms, and the exterior derivative operator, d.

SOLUTION: We have

$$\begin{aligned} d\phi &= \omega_{\operatorname{grad}(\phi)}, \\ d\omega_F &= \omega^{\operatorname{curl}(F)}, \\ d\omega^F &= \operatorname{div}(F)\,dx \wedge dy \wedge dz. \end{aligned}$$

7. (a) State Stokes' theorem in terms of differential forms.

SOLUTION: If S is a k-chain in $\mathbb{R}^n$, and ω is a $(k-1)$-form in $\mathbb{R}^n$, then

$$\int_{\partial S} \omega = \int_S d\omega.$$

(b) Starting with $\omega \in \Omega^i\mathbb{R}^3$ for each of $i = 0, 1, 2$, give a classical/physical interpretation of Stokes' theorem.

SOLUTION: If $\omega \in \Omega^0\mathbb{R}^3$, then ω is just a real-valued function on $\mathbb{R}^3$. Say $\omega = \phi$. Then $d\omega = d\phi = \omega_{\nabla\phi}$, the flow form for the gradient of ϕ.

Given a curve $C : [0,1] \to \mathbb{R}^3$, Stokes' theorem says

$$\int_C \nabla\phi \cdot \vec{t} = \int_C d\phi = \int_{\partial C} \phi = \phi(C(1)) - \phi(C(0)).$$

The flow of a gradient vector field is given by the change in potential.

If $\omega \in \Omega^1\mathbb{R}^3$, then we can write $\omega = \omega_F$ for some vector field F on $\mathbb{R}^3$. Let $S : [0,1]^2 \to \mathbb{R}^3$ be a surface in $\mathbb{R}^3$. Then Stokes' theorem says

$$\int_S (\nabla \times F) \cdot \vec{n} = \int_S \omega^{\nabla\times F} = \int_S d\omega_F = \int_{\partial S} \omega_F = \int_{\partial S} F \cdot \vec{t}.$$

The flux of the curl of a vector field through a surface is the circulation of the vector field along the boundary of the surface.

If $\omega \in \Omega^2\mathbb{R}^3$, then we can write $\omega = \omega^F$ for some vector field F on $\mathbb{R}^3$. Let $V : [0,1]^3 \to \mathbb{R}^3$ be a parametrized solid in $\mathbb{R}^3$. Then Stokes' theorem says

$$\int_V (\nabla \cdot F)\, dV = \int_V \nabla \cdot F\, dx \wedge dy \wedge dz = \int_V d\omega^F = \int_{\partial V} \omega^F = \int_{\partial V} F \cdot \vec{n}.$$

The integral of the divergence of F over a solid is the flux of F through the boundary of the solid.

8. What does the fact that $d^2 = 0$ say in terms of grad, curl, and div?

SOLUTION: Let ϕ be a real-valued function on $\mathbb{R}^3$. Then

$$0 = d^2\phi = d(\omega_{\mathrm{grad}(\phi)}) = \omega^{\mathrm{curl}(\mathrm{grad}(\phi))}.$$

Hence, $\mathrm{curl}(\mathrm{grad}(\phi)) = \nabla \times (\nabla\phi) = 0$.

Similarly, if F is a vector field in $\mathbb{R}^3$, then

$$0 = d^2\omega_F = d(\omega^{\mathrm{curl}(F)}) = \mathrm{div}(\mathrm{curl}(F))\, dx \wedge dy \wedge dz.$$

Hence, $\mathrm{div}(\mathrm{curl}(F)) = \nabla \cdot (\nabla \times F) = 0$.

9. Let ω be a k-form. When is it true that $d\omega = 0$ implies there exists a $(k-1)$-form, λ such that $\omega = d\lambda$? If the condition holds, what is the implication for grad, for curl, and for div?

SOLUTION: A subset U of $\mathbb{R}^n$ is *simply connected* if each closed curve in U can be shrunk to a point without leaving U. The plane with the origin removed is not simply connected. If the domain of ω is simply connected, for example, if the domain is all of $\mathbb{R}^n$, then $d\omega = 0$ implies the existence of a form λ such that $\omega = d\lambda$.

So suppose we have a vector field F defined on a simply connected subset of $\mathbb{R}^3$ (for instance, defined on all of $\mathbb{R}^3$). Then if $\mathrm{curl}(F) = 0$, we know that F is a gradient vector field, i.e., F has a potential. (Reason: $\mathrm{curl}(F) = 0$ implies $0 = \omega^{\mathrm{curl}(F)} = d\omega_F$, which implies $\omega = d\phi$ for some 0-form ϕ.) Similarly, if $\mathrm{div}(F) = 0$, then F has a vector potential, i.e., $F = \mathrm{curl}(G)$ for some vector field G. (Reason: $\mathrm{div}(F) = 0$ implies $0 = \mathrm{div}(F)\, dx \wedge dy \wedge dz = d\omega^F$, which implies $\omega^F = d\lambda$ for some 1-form λ. We can then write $\lambda = \omega_G$ for some vector field G. It follows that $F = \mathrm{curl}(G)$.)