Math 212

* Review HW.

Reformulation of Maxwell's equations

Hodge $*$ operator

There exist an isomorphism of vector spaces

$$* : \Omega^k \mathbb{R}^n \longrightarrow \Omega^{n-k}$$

for each $k = 0, 1, \ldots, n$. It is determined by

$$(dx_{i_1} \wedge \cdots \wedge dx_{i_k}) \wedge \left[* (dx_{i_1} \wedge \cdots \wedge dx_{i_k}) \right] = dx_1 \wedge \cdots \wedge dx_n .$$

Example

$$*dx = dy\wedge dz, \qquad *dy = -dx\wedge dz, \qquad *dz = dx\wedge dy.$$

- If F is a vector field in $\mathbb{R}^n$ and $\omega_F = \sum F_i\, dx_i$ is its flow form, then $*\omega_F = \omega^F$ is the flux form for F.

- **Lemma** For $*$ on $\mathbb{R}^n$ (with the usual metric), we have

$$** = (-1)^{k(n-k)}\, id$$ where $id : \Omega^k\mathbb{R}^n \longrightarrow \Omega^k\mathbb{R}^n$ is the identity mapping $id(\omega) = \omega$.

Pf/ Exercise. $\square$

Def $\delta = * d *$

$$\Omega^k \mathbb{R}^n \xrightarrow{*} \Omega^{n-k} \mathbb{R}^n \xrightarrow{d} \Omega^{n-k+1} \mathbb{R}^n \xrightarrow{*} \Omega^{k-1} \mathbb{R}^n$$

Def. If ω_F is the flow form for a vector field F in $\mathbb{R}^n$, then

$$\operatorname{div} F = * d * \omega_F \in \Omega^0 \mathbb{R}^n.$$

Hodge $*$ on Minkowski space

Consider $\mathbb{R}^4$ with coordinates t, x, y, z and metric

$$\langle u, v \rangle = u^t \begin{bmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix} v = -u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4$$

The Hodge $*$ operator depends on the metric in general in a way that will not be specified here, but for the above metric, we have

- $*\left(F_1\, dx\wedge dy\wedge dz + F_2\, dt\wedge dy\wedge dz + F_3\, dt\wedge dx\wedge dz + F_4\, dt\wedge dx\wedge dy\right)$

  $= -\left(F_1\, dt + F_2\, dx + F_3\, dy + F_4\, dz\right)$

- $*\left(F_1\, dt\wedge dx + F_2\, dt\wedge dy + F_3\, dt\wedge dz + F_4\, dx\wedge dy + F_5\, dx\wedge dz + F_6\, dy\wedge dz\right.$

  $= -F_1\, dy\wedge dz + F_2\, dx\wedge dz + F_3\, dx\wedge dy + F_4\, dt\wedge dz - F_5\, dt\wedge dy + F_6\, dt\wedge dx$

- $*1 = dt\wedge dx\wedge dy\wedge dz$

- $*^2 = (-1)^{k(n-k)+1}\, id$ for $* : \Omega^k \mathbb{R}^4 \to \Omega^{n-k}\mathbb{R}^4$ with the Minkowski metric

For a vector field $F = F(t,x,y,z)$ in $\mathbb{R}^3$ depending on time, write $\omega_F = F_1\,dx + F_2\,dy + F_3\,dz$ and $\omega^F = F_1\,dy\wedge dz - F_2\,dx\wedge dz + F_3\,dx\wedge dy$.

Define

$$\alpha = \omega^{\mathbb{B}} + \omega_{\mathbb{E}} \wedge dt.$$

$$\beta = -\rho\,dt + \omega_{\mathbb{J}}$$

Maxwell with units taken to absorb μ_0 and ε_0.

Prop. (1) $\delta\alpha = \beta$ iff $\left(\nabla\cdot\mathbb{E} = \rho \text{ and } \nabla\times\mathbb{B} = \mathbb{J} + \frac{\partial\mathbb{E}}{\partial t}\right)$

(2) $d\alpha = 0$ iff $\left(\nabla\times\mathbb{E} = -\frac{\partial\mathbb{B}}{\partial t} \text{ and } \nabla\cdot\mathbb{B} = 0\right)$

Pf/ (1) Direct calculation gives $*\alpha = \omega^{\mathbb{E}} - \omega_{\mathbb{B}}\wedge dt$, and

$\delta\alpha = *d*\alpha = -\nabla\cdot\mathbb{E}\,dt - \omega_{\frac{\partial\mathbb{E}}{\partial t}} + \omega_{\nabla\times\mathbb{B}}$. Therefore $\delta\alpha = \beta$ iff

$\nabla\cdot\mathbb{E} = \rho$ and $-\frac{\partial\mathbb{E}}{\partial t} + \nabla\times\mathbb{B} = \mathbb{J}$.

(2) $d\alpha = d\left[\omega^{\mathbb{B}} + \omega_{\mathbb{E}} \wedge dt\right] = (dB_1) \wedge dy \wedge dz - (dB_2) \wedge dx \wedge dz + (dB_3) \wedge dx \wedge dy$

$$+ \, d\omega_{\mathbb{E}} \wedge dt$$

Note:

$$dB_1 = \frac{\partial B_1}{\partial t} dt + \frac{\partial B_1}{\partial x} dx + \frac{\partial B_1}{\partial y} dy + \frac{\partial B_1}{\partial z} dz$$

$$= \omega^{\frac{\partial \mathbb{B}}{\partial t}} \wedge dt + \nabla \cdot \mathbb{B} \, dx \wedge dy \wedge dz$$

$$+ \, \omega^{\nabla \times \mathbb{E}} \wedge dt$$

$$= 0$$

iff $\frac{\partial \mathbb{B}}{\partial t} + \nabla \times \mathbb{E} = 0$ and $\nabla \cdot \mathbb{B} = 0$. $\square$