Math 212 Review Problem Solutions

1. Under the influence of the force $F = (e^{-y} - ze^{-x})\vec{i} + (e^{-z} - xe^{-y})\vec{j} + (e^{-x} - ye^{-z})\vec{k}$ a particle moves along the path

$$x = \frac{1}{\ln 2}\ln(1+t), \qquad y = \sin\frac{\pi t}{2}, \qquad z = \frac{1-e^{t}}{1-e}$$

for $t \in [0,1]$. How much work does the force do on the particle?

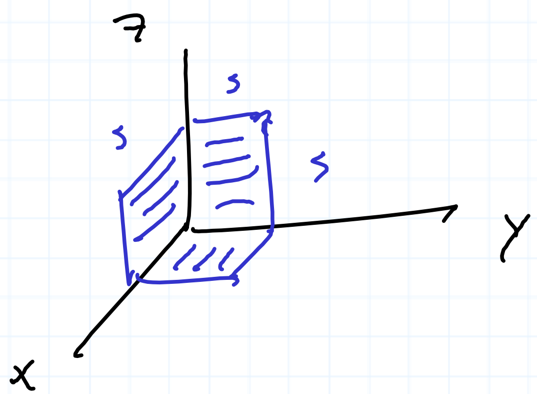
Solution/ By inspection, $\varphi(x,y,z) = ze^{-x} + xe^{-y} + ye^{-z}$ is a potential for F. Therefore, if C is the path,

$$\int_C F\cdot\vec{t} = \varphi(c(1)) - \varphi(c(0)) = \varphi(1,1,1) - \varphi(0,0,0)$$

$$= 3/e.$$

2. Find the flux $\iint_S F \cdot \vec{n}$.

(a) $F = (x, y, z)$; S is formed by the 3 squares of side length 5 meeting at the origin, pictured below:



Solution/ We need to choose an orientation. We'll take the flux of F coming out of the screen/paper. We compute the flux through each side:

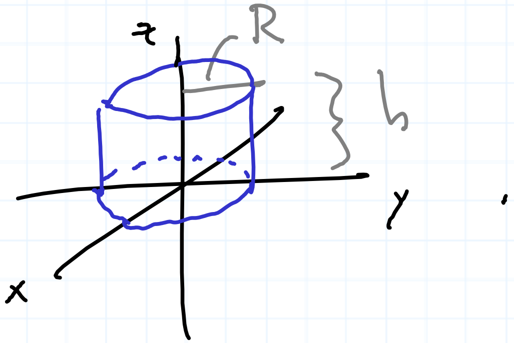
$x = 0$: $F \cdot \vec{n} = (0, y, z) \cdot (1, 0, 0) = 0$, so flux $= 0$ through this side

$y=0$: $F\cdot\vec{n} = (x,0,z)\cdot(0,1,0) = 0$, so flux $=0$.

$z=0$: $F\cdot\vec{n} = (x,y,0)\cdot(0,0,1) = 0$, so flux $=0$.

Therefore, the total flux is 0.

(b) $F = (x\vec{i} + y\vec{j})(\ln(x^2+y^2))$; S is the cylinder of radius R and height h pictured below:



Solution/ There is no flux through the top and bottom since $F\cdot\vec{n} = 0$ there. On the side, F and $\vec{n}$ point in the same direction, so $F\cdot\vec{n} = |F| = R\ln R^2 = 2R\ln R$. The flux is the area of the side, weighted by the normal component of F:

$$(2R\ln R)(2\pi Rh) = 4\pi R^2 h \ln R.$$

(c) $F = (x,y,z)\, e^{-(x^2+y^2+z^2)}$; S is the sphere radius R centered at the origin.

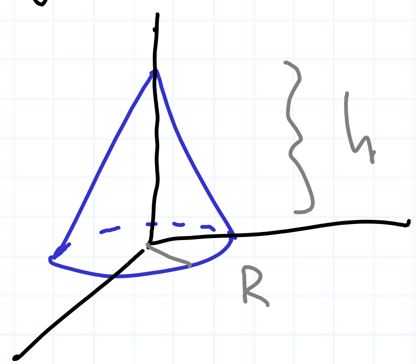
Solution/ F is radial, so $F\cdot\vec{n} = |F| = Re^{-R^2}$ and the flux is $Re^{-R^2}\cdot \text{area}(S) = Re^{-R^2}(4\pi R^2) = 4\pi R^3 e^{-R^2}$.

(d) $F = f(x)\,\hat{i}$; $S = \partial B_s$ where $B_s = [0,s]^3$.

Solution/ $F\cdot\hat{n} = 0$ on each side of the cube except the sides where $x=0$ and $x=s$. Taking outward pointing normals, we have $F(0,y,z)\cdot(-1,0,0) = -f(0)$ and $F(s,y,z)\cdot(1,0,0) = f(s)$.

So the total flux is $(f(s)-f(0))s^2$.

(e) $F = (x, 2y, 3z)$; S is the right circular cone of radius R at the base and height h, centered at the origin:



Solution / By the divergence theorem,

$$\iint_S F \cdot \vec{n} = \iiint_{\text{cone}} \operatorname{div} F = \iiint_{\text{cone}} 6 = 6 \operatorname{vol}(\text{cone}) = 2\pi R^2 h.$$

3. Consider the parametrized surface $S(u,v) = (\underset{x}{u^2 \cos v}, \underset{y}{u^2 \sin v}, \underset{z}{u})$

a) Find the unit normal to S at $u=1$, $v=0$ using cross products.

Solution/ $S_u = (2u\cos v, 2u\sin v, 1)$

$S_v = (-u^2\sin v, u^2\cos v, 0)$

$S_u \times S_v = (-u^2\cos v, -u^2\sin v, 2u^3)$

At $u=1, v=0$, we have $\vec{n} = \dfrac{(S_u\times S_v)(1,0)}{|(S_u\times S_v)(1,0)|} = \dfrac{(-1,0,2)}{\sqrt{5}}$.

b) Describe the tangent plane to S at $u=1, v=0$ by an equation of the form $ax+by+cz=d$ for some $a,b,c,d\in\mathbb{R}$

Solution/ Since the tangent plane is normal to $\vec{n} = (-1,0,2)/\sqrt{5}$, we can take $(a,b,c) = \lambda\vec{n}$ for any $\lambda\neq 0$, say $\lambda=\sqrt{5}$.

Then the equation for the tangent plane has the form

$$-x+2z=d.$$

Since the plane passes through $S(1,0) = (1,0,1)$, we get

$$-1 + 2\cdot 1 = d \Rightarrow d = 1.$$

So an equation for the tangent plane is

$$-x + 2z = 1.$$

c) Describe the image of S by an equation $f(x,y,z) = 0$
(So $\text{image}(S) = \{(x,y,z) \in \mathbb{R}^3 : f(x,y,z) = 0\}$.)

Solution/ Let $x = u^2 \cos v$, $y = u^2 \sin v$, $z = u$. Then eliminating u and v gives

$$x^2 + y^2 = z^4.$$

Take $f(x,y,z) = x^2 + y^2 - z^4.$

d) Use the gradient of f (from part (c)) to verify your answer to part (a).

Solution / $\nabla f = (2x, 2y, -4z^3) \Rightarrow$

$$\nabla f(S(1,0)) = \nabla f(1,0,1) = (2, 0, -4).$$

The image of S is the level set $f(x,y,z) = 0$. The gradient, ∇f, is perpendicular to the level set, so

$$\vec{n} = \frac{(2,0,-4)}{\|(2,0,-4)\|} = \frac{(1,0,-2)}{\sqrt{5}}.$$

$x - 2z = d$ passing through $(1,0,1)$
$\Rightarrow d = -1$ Equation: $x - 2z = -1$
$\Rightarrow -x + 2z = 1$, as before.

Note: This is the same as in (a) but pointing in the other direction. (We could have taken $f(x,y,z) = -x^2 - y^2 + z^4$ to get an exact match with (a).)

4. Evaluate $\iint_S \frac{1}{1+4(x^2+y^2)}\, dS$ where S is the portion of the paraboloid $z = x^2+y^2$ between $z=0$ and $z=1$.

Solution/ Parametrize S:

$$\Phi(\theta, z) = (\sqrt{z}\cos\theta, \sqrt{z}\sin\theta, z),$$

$$0 \le \theta \le 2\pi, \qquad 0 \le z \le 1.$$

Then

$$\Phi_\theta = (-\sqrt{z}\sin\theta, \sqrt{z}\cos\theta, 0)$$

$$\Phi_z = \left(\frac{1}{2\sqrt{z}}\cos\theta, \frac{1}{2\sqrt{z}}\sin\theta, 1\right)$$

$$\Phi_\theta \times \Phi_z = \left(\sqrt{z}\cos\theta, \sqrt{z}\sin\theta, -\frac{1}{2}\right)$$

$$|\Phi_\theta \times \Phi_z| = \sqrt{z+\frac{1}{4}} = \frac{\sqrt{1+4z}}{2}.$$

The integral becomes

$$\iint_{\Phi} \frac{1}{1+4(x^2+y^2)} = \int_0^{2\pi}\int_0^1 \overbrace{\frac{1}{1+4z}}^{\Phi\circ f} \cdot \overbrace{\frac{\sqrt{1+4z}}{2}}^{|\Phi_\theta \times \Phi_z|} \, dz\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 \frac{1}{2}(1+4z)^{-\frac{1}{2}}\, dz\, d\theta$$

$$= \frac{1}{4}\int_0^{2\pi}\left[(1+4z)^{\frac{1}{2}}\Big|_0^1\right] d\theta$$

$$= \frac{1}{4}\int_0^{2\pi}\left(\sqrt{5}-1\right) d\theta$$

$$= \frac{\pi}{2}\left(\sqrt{5}-1\right).$$