1. Find the area of the triangle in the plane with vertices $(0,0)$, $(1,0)$, and $(1,1)$, weighted by the function $f(x,y) = 8xy$.

   SOLUTION: $\int_{x=0}^{1} \int_{y=0}^{x} 8xy = \int_{x=0}^{1} 4x^3 = 1$.

2. Find the flow of the vector field $F(x,y) = (x, y^2)$ about the triangle in the plane with vertices $(0,0)$, $(1,0)$, and $(0,1)$, oriented counterclockwise.

   SOLUTION: The boundary of the triangle is parametrized by the three curves

   $$C_1(t) = (t,0), \quad C_2(t) = (1-t,t), \quad C_3(t) = (0,1-t),$$

   each with $0 \le t \le 1$. The total flow is

$$\begin{aligned}
\int_{\partial\Delta} F \cdot \vec{t} &= \sum_{i=1}^{3} \int_{C_i} F \cdot \vec{t} \\
&= \sum_{i=0}^{3} \int_0^1 F(C_i(t)) \cdot C_i'(t) \\
&= \int_0^1 F(t,0) \cdot (1,0) + \int_0^1 F(1-t,t) \cdot (-1,1) + \int_0^1 F(0,1-t) \cdot (0,-1) \\
&= \int_0^1 (t,0) \cdot (1,0) + \int_0^1 (1-t,t^2) \cdot (-1,1) + \int_0^1 (0,(1-t)^2) \cdot (0,-1) \\
&= \int_0^1 t + \int_0^1 t^2 + t - 1 + \int_0^1 (-t^2 + 2t - 1) \\
&= \int_0^1 4t - 2 \\
&= 0.
\end{aligned}$$

Another solution, using Stokes' theorem:

$$\begin{aligned}\int_{\partial\Delta} F\cdot\vec{t} &= \int_{\partial\Delta}\omega_F\\ &= \int_{\Delta} d\,\omega_F\\ &= \int_{\Delta} d(x\,dx + y^2y)\\ &= \int_{\Delta} dx\wedge dx + 2y\,dy\wedge dy\\ &= 0.\end{aligned}$$

Yet another solution, again using Stokes': the vector field F has a potential function $\phi = x^2/2 + y^3/3$. The flow of F along the curve is given by the change in potential. Since the curve is closed, this change in potential is 0.

3. Find the flux of $F(x,y,z) = (z,4y,x)$ through the surface

$$\begin{aligned} S\colon [0,1]^2 &\to \mathbb{R}^3\\ (u,v) &\mapsto (v,uv,u).\end{aligned}$$

SOLUTION: First note that

$$S_u\times S_v = \begin{pmatrix} i & j & k\\ 0 & v & 1\\ 1 & u & 0\end{pmatrix} = (-u,1,-v).$$

Then,

$$\begin{aligned}\int_S F\cdot\vec{n} &= \int_{[0,1]^2} F(S(u,v))\cdot(S_u\times S_v)\\ &= \int_{[0,1]^2} F(v,uv,u)\cdot(-u,1,-v)\\ &= \int_{[0,1]^2}(u,4uv,v)\cdot(-u,1,-v)\\ &= \int_{u=0}^{1}\int_{v=0}^{1} -u^2+4uv-v^2\\ &= \int_{u=0}^{1} -u^2+2u-\frac{1}{3}\\ &= \frac{1}{3}.\end{aligned}$$

4. What is the area of the surface parametrized by

$$\begin{aligned} S\colon [0,1]^2 &\to \mathbb{R}^3 \\ (u,v) &\mapsto (v, uv, u) \end{aligned}$$

weighted by the function $f(x,y,z) = xz$?

SOLUTION: From the previous problem, we have $S_u \times S_v = (-u, 1, -v)$. Therefore,

$$\begin{aligned} \int_S f &= \int_{[0,1]^2} f(S(u,v))\,|(S_u \times S_v)| \\ &= \int_{[0,1]^2} f(v, uv, u)\,\sqrt{u^2+1+v^2} \\ &= \int_{u=0}^{1}\int_{v=0}^{1} vu\,\sqrt{u^2+1+v^2} \\ &= \int_{u=0}^{1}\left(\frac{1}{3}u\sqrt{u^2+1+v^2}\right)|_{v=0}^{1} \\ &= \int_{u=0}^{1}\frac{1}{3}u\sqrt{u^2+2}-\frac{1}{3}u\sqrt{u^2+1} \\ &= \left(\frac{1}{15}(2+u^2)^{5/2}-\frac{1}{15}(1+u^2)^{5/2}\right)_{u=1}^{1} \\ &= \frac{1}{15}(3^{5/2}-1). \end{aligned}$$

5. Find the flux of $F(x,y,z) = (y^2, \cos(xz), x^2+3z)$ through the unit sphere centered at the origin.

SOLUTION: First note that

$$\operatorname{div} F = D_1 y^2 + D_2\cos(xz) + D_3(x^2+3z) = 3.$$

Then, by Stokes' theorem,

$$\int_{\text{sphere}} F\cdot\vec{n} = \int_{\text{solid ball}} \operatorname{div} F = 3(4\pi/3) = 4\pi.$$

6. What is the length of the curve $C(t) = (t, t^2)$ for $0 \le t \le 1$, weighted by $f(x,y) = x$?

SOLUTION:

$$\begin{aligned}\int_C f &= \int_0^1 f(C(t))\,|C'(t)|\,dt \\ &= \int_0^1 f(t,t^2)\,|(1,2t)|\,dt \\ &= \int_0^1 t\sqrt{1+4t^2}\,dt \\ &= \frac{1}{12}(1+4t^2)^{3/2}|_{t=0}^{1} \\ &= \frac{1}{12}(5^{3/2}-1).\end{aligned}$$

7. What is the flow of $F(x,y) = (-y, y^3)$ around the circle $C(t) = (\cos(t), \sin(t))$ for $0 \le t \le 2\pi$?

SOLUTION:

$$\begin{aligned}\int_C F\cdot\vec{t} &= \int_0^{2\pi} F(C(t))\cdot C'(t) \\ &= \int_0^{2\pi} F(\cos(t),\sin(t))\cdot(-\sin(t),\cos(t))\,dt \\ &= \int_0^{2\pi} (-\sin(t),\sin^3(t))\cdot(-\sin(t),\cos(t))\,dt \\ &= \int_0^{2\pi} \sin^2(t)+\sin^3(t)\cos(t)\,dt \\ &= \int_0^{2\pi} \frac{1-\cos(2t)}{2}+\sin^3(t)\cos(t)\,dt \\ &= \left(\frac{t}{2}-\frac{\sin(2t)}{4}+\frac{1}{4}\sin^4(t)\right)|_{t=0}^{2\pi} \\ &= \pi.\end{aligned}$$

It is a lot less work to use Stokes' theorem, instead:

$$\begin{aligned}
\int_C F \cdot \vec{t} &= \int_C \omega_F \\
&= \int_{\text{disk}} d\omega_F \\
&= \int_{\text{disk}} d(-y\,dx + y^3\,dy) \\
&= \int_{\text{disk}} -dy \wedge dx + 3y^2\,dy \wedge dy \\
&= \int_{\text{disk}} dx \wedge dy \\
&= \text{area}(\text{disk}) \\
&= \pi.
\end{aligned}$$