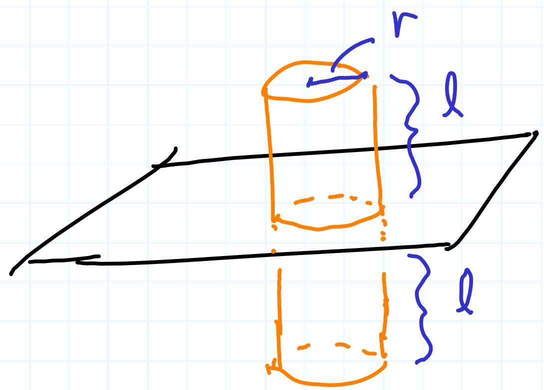
Solutions to problems I - IV

I. Think of the xy-plane as an infinite slab of charge with constant density λ (charge per unit area). Making reasonable symmetry assumptions, calculate the electric field, $\vec{E}$, at points off of the plane.

Solution/



Consider a cylinder through the plane, as shown. We've chosen a circular cylinder, but that's not important. By symmetry, we'll assume $\vec{E}$ has no flux through the sides of the cylinder and that $\vec{E}$ is perpendicular to the top and bottom of the cylinder (and constant there). Using $\nabla \cdot \mathbf{E} = \rho/\varepsilon_0$, we get

$$\text{flux of } \mathbf{E} \text{ through cylinder} = \oiint_{\text{cylinder}} \mathbf{E} = \iiint \nabla \cdot \mathbf{E}$$

$$= \iiint \rho/\varepsilon_0 = \text{enclosed charge}/\varepsilon_0 .$$

The flux is the area of the top and bottom weighted by the normal component of $\mathbf{E}$:

$$\text{flux} = |\mathbf{E}|(\underbrace{\pi r^2}_{\text{top}} + \underbrace{\pi r^2}_{\text{bottom}}) = 2|\mathbf{E}|\pi r^2,$$

The enclosed charge is

$$(\text{area of plate enclosed}) \times \text{density} = \pi r^2 \lambda .$$

Therefore,

$$2|\mathbb{E}|\pi r^2 = \pi r^2 \lambda / \varepsilon_0 \Rightarrow |\mathbb{E}| = \frac{\lambda}{2\varepsilon_0}$$

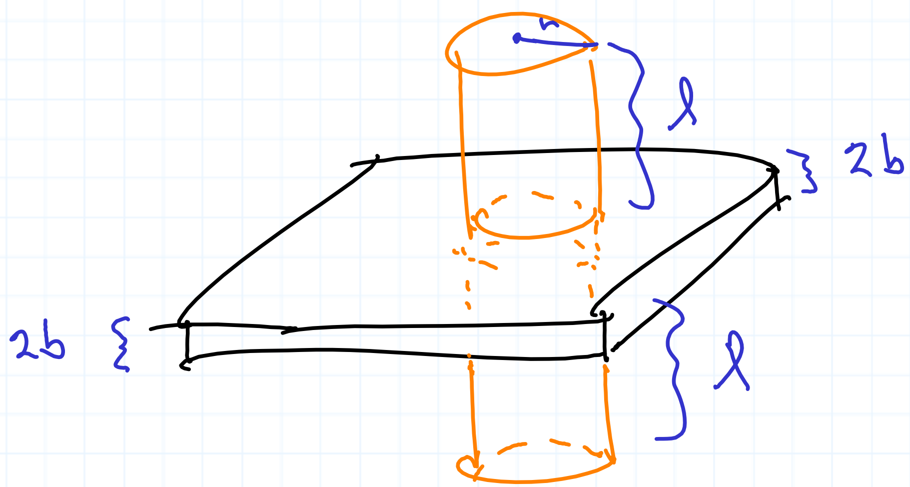
and the direction is perpendicular to the plate (pointing away from the plate). $\square$

II Consider the distribution of charge in $\mathbb{R}^3$ having density

$$\rho(x,y,z) = \begin{cases} \rho_0 & \text{if } -b \le z \le b \\ 0 & \text{if } |z| > b \end{cases}$$

for some constants ρ_0 and $b > 0$. Making reasonable symmetry assumptions, calculate the electric field, $\mathbb{E}$, at points $(x,y,z) \in \mathbb{R}^3$ with $|z| > b$.

Solution / Proceed as in problem I:



Flux through cylinder a distance l from plate $= |\mathbb{E}(l)|(\underbrace{\pi r^2}_{\text{top}} + \underbrace{\pi r^2}_{\text{bottom}}) = 2|\mathbb{E}(l)|\pi r^2$

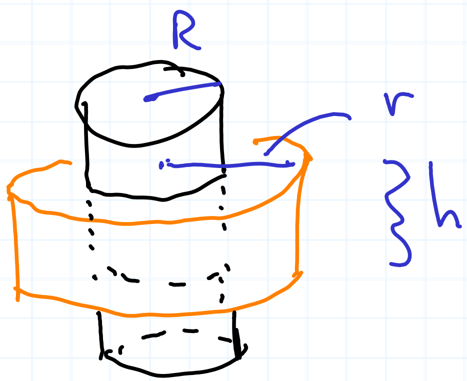
$= \text{enclosed charge}/\varepsilon_0$

$= 2b\pi r^2 \rho_0/\varepsilon_0$

$\Longrightarrow |\mathbb{E}(l)| = b\rho_0/\varepsilon_0$ (and the direction is perp. + away from the plate). $\square$

III Consider a hollow circular cylinder of infinite length and radius R centered along the z-axis. Suppose a uniform charge on the cylinder with (constant) density λ (Coulombs/meter^2). Making reasonable symmetry assumptions, find the electric field caused by the cylinder.

Solution/



component perpendicular to side

area of side

Flux through outer cylinder $= |E(r)|\, 2\pi r h$

$= \text{enclosed charge} / \varepsilon_0$

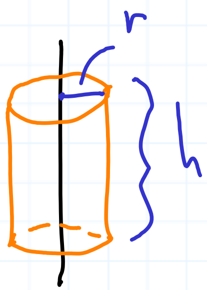
$= 2\pi R h \lambda / \varepsilon_0$

$$\Longrightarrow \quad |E(r)| = \left(\frac{R\lambda}{\varepsilon_0}\right)\frac{1}{r}$$ with direction radially outward from z-axis.

For points inside the cylinder ($r < R$) there is no enclosed charge in our test surface. Hence, $E = 0$ inside the cylinder. $\square$

IV. Consider the z-axis to be an infinitely long wire with uniform charge density λ (Coulombs/meter). Making reasonable symmetry assumptions, determine the electric field caused by the wire. What if the wire is replaced by a cylinder of charge with density $\rho(r) = \rho_0 e^{-r/b}$ where r is the distance from the z-axis?

Solution/



$$\text{Flux} = |E(r)|\, 2\pi r h$$
$$= \text{enclosed charge}/\varepsilon_0 = \lambda h/\varepsilon_0$$

$$\Rightarrow \quad |E(r)| = \left(\frac{\lambda}{2\pi\varepsilon_0}\right)\frac{1}{r}$$ with direction radially out from z-axis.

For a cylinder of charge with density $\rho(r) = \rho_0 e^{-r/b}$, the charge enclosed in a cylinder of radius r and height h is

$$\iiint_{\substack{\text{closed}\\\text{cylinder}}} \rho_0 e^{-r/b} \underset{\substack{\text{cylindrical}\\\text{coordinates}}}{=} \int_{\theta=0}^{2\pi}\int_{z=-\frac{h}{2}}^{\frac{h}{2}}\int_{t=0}^{r} t\,\rho_0 e^{-t/b}$$

stretching factor

$$= 2\pi h \rho_0 \int_0^r t e^{-t/b}\,dt$$

By parts:

$u = t \qquad du = dt$

$dv = e^{-t/b}dt \qquad v = -be^{-t/b}$

$$= 2\pi h \rho_0 \left[-bte^{-t/b}\Big|_0^r + \int_0^r b e^{-t/b}\,dt\right]$$

$$= 2\pi h \rho_0 \left[-bre^{-r/b} + \left(-b^2 e^{-t/b}\Big|_0^r\right)\right]$$

$$= 2\pi h \rho_0 \left[-bre^{-r/b} - b^2 e^{-r/b} + b^2\right]$$

The flux of $\mathbb{E}$ through the cylinder is $|\mathbb{E}(r)|\, 2\pi r h$.

By Gauss' law,

$$|\mathbb{E}(r)| = \left(b\rho_0 \left[-(r+b)e^{-r/b} + b \right] / \varepsilon_0 \right) \left(\frac{1}{r} \right). \quad \square$$