Math 212

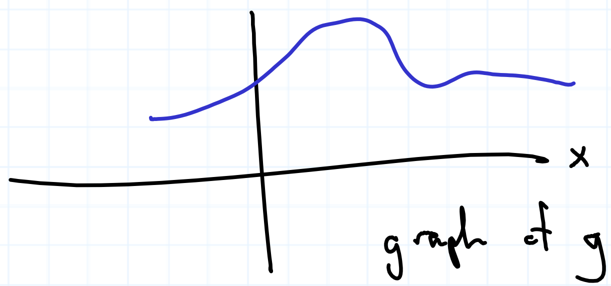
Quiz: See webpage

Questions about HW.

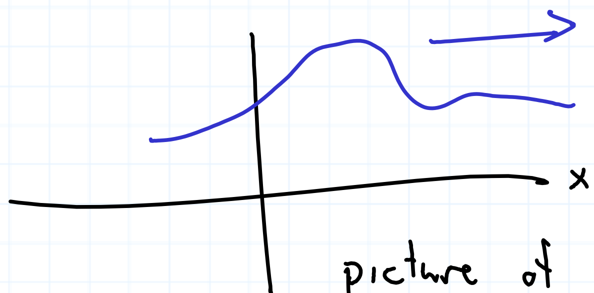
Maxwell's equations & electromagnetic waves

Wave equation: Let $g : \mathbb{R} \to \mathbb{R}$. Fix $v \in \mathbb{R}$ and define

$$f(x,t) = g(x - vt),$$



graph of g



picture of $f(x,t)$: traveling wave

We have $\dfrac{\partial^2 f}{\partial t^2}(x,t) = v^2 g''(x-vt) = v^2 \dfrac{\partial^2 f}{\partial x^2} \implies \dfrac{\partial^2 f}{\partial x^2} \overset{\star}{=} \dfrac{1}{v^2} \dfrac{\partial^2 f}{\partial t^2}$

The most general solution to $\star$ is $f(x,t) = g(x-vt) + h(x+vt)$

for some functions g and h (waves traveling in opposite directions).

Equation ✡ is the wave equation in one dimension.

Lemma For any vector field $F : \mathbb{R}^3 \to \mathbb{R}^3$,

$$\underbrace{\nabla \times (\nabla \times F)}_{\text{curl of curl}} = \underbrace{\nabla(\nabla \cdot F)}_{\text{gradient of divergence}} - \underbrace{\nabla^2}_{\text{Laplacian}} F.$$

Laplacian: $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

Pf/ HW.

Consider Maxwell's equations in empty space ($\mathbb{J} = 0$, $\rho = 0$):

$$\nabla \cdot \mathbb{E} = 0, \qquad \nabla \times \mathbb{E} = -\frac{\partial \mathbb{B}}{\partial t}$$

$$\nabla \cdot \mathbb{B} = 0, \qquad \nabla \times \mathbb{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbb{E}}{\partial t}.$$

By the lemma,

$$\nabla^2 \mathbb{E} = \nabla(\nabla \cdot \mathbb{E}) - \nabla \times (\nabla \times \mathbb{E})$$

$$\overset{\text{Maxwell}}{=} \nabla(0) - \nabla \times \left(-\frac{\partial \mathbb{B}}{\partial t}\right)$$

$$= \frac{\partial}{\partial t}(\nabla \times \mathbb{B})$$

$$\overset{\text{Maxwell}}{=} \mu_0 \varepsilon_0 \frac{\partial^2 \mathbb{E}}{\partial t^2}$$

So $\mathbb{E}$ satisfies the wave equation

$$\nabla^2 \mathbb{E} = \frac{1}{v^2} \frac{\partial^2 \mathbb{E}}{\partial t^2}$$

with $v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$. It turns out that $\frac{1}{\sqrt{\mu_0 \varepsilon_0}} = c$, the speed of light.