Math 212

Coulomb's Law from Maxwell's equations

$$\nabla \cdot \mathbb{E} = \frac{1}{\varepsilon_0}\rho$$

$$\nabla \cdot \mathbb{B} = 0$$

$$\nabla \times \mathbb{E} = -\frac{\partial \mathbb{B}}{\partial t}$$

$$\nabla \times \mathbb{B} = \mu_0 \mathbb{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbb{E}}{\partial t}$$

* Suppose you have a small ball of charge with density ρ, and suppose its electric field in radially symmetric about the center of the ball. If B is a large ball of radius r about the ball of charge, then Maxwell says

$$\iint_{\partial B} \mathbb{E} \cdot \vec{n} = \iiint_B \nabla \cdot \mathbb{E} = \frac{1}{\varepsilon_0} \iiint_B \rho = q/\varepsilon_0$$

where q = total amt. of charge

On the other hand, radial symmetry implies $\iint_{\partial B} \mathbb{E} \cdot \vec{n} =$ $|\mathbb{E}|$ (surface area of ∂B) $= |\mathbb{E}|\, 4\pi r^2$. Hence, $|\mathbb{E}| = \frac{q}{4\pi \varepsilon_0 r^2}$

In this way we recover Coulomb's law.

Conservation of charge

$\mathbb{J} = \underbrace{\text{current}}_{\text{Amperes: } A = \frac{\text{Coulombs}}{\text{sec}}}\ \text{density} = \dfrac{\text{current}}{\text{area perpendicular to flow}} = \dfrac{\text{charge}}{\text{time} \cdot \text{area}}$

The current crossing through a surface S is $\iint_S \mathbb{J} \cdot \vec{n}$.

Let B_r be a ball of radius r centered at $p \in \mathbb{R}^3$. Then if charge is conserved,

$$\underbrace{\iint_{\partial B_r} \mathbb{J} \cdot \vec{n}}_{\text{charge leaving } B_r \text{ per unit time}} = -\frac{\partial}{\partial t} \iiint_{B_r} \rho \iff \frac{1}{\mathrm{vol}(B_r)} \iint \mathbb{J} \cdot \vec{n} = -\frac{\partial}{\partial t} \frac{1}{\mathrm{vol}(B_r)} \iiint_{B_r} \rho$$

take the limit as $r \to 0$:

$$\nabla \mathbb{J}(p) \qquad -\frac{\partial \rho}{\partial t}(p)$$

rho, p

Thus, $\nabla \mathbb{J} = -\frac{\partial \rho}{\partial t}$ is the statement that charge is conserved.

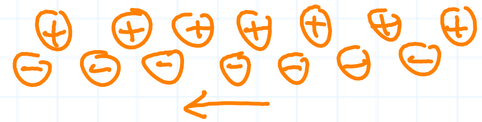
$0 = d^2\omega_{B} = d(\omega^{\nabla\times B}) = \nabla\cdot(\nabla\times B)\, dx\wedge dy\wedge dz$

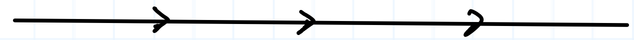
Conservation of charge follows from Maxwell's equations:

$$0 \overset{d^2=0}{=} \nabla\cdot(\nabla\times B) \overset{\text{Max.}}{=} \nabla\cdot\left(\mu_0 J + \mu_0\varepsilon_0 \frac{\partial E}{\partial t}\right) = \mu_0 \nabla\cdot J + \mu_0\varepsilon_0 \frac{\partial}{\partial t}\nabla\cdot E$$

$$\overset{\text{Max.}}{=} \mu_0 \nabla\cdot J + \mu_0 \frac{\partial}{\partial t}\rho \implies \nabla\cdot J = -\frac{\partial \rho}{\partial t}.$$

Magnetic field about a wire



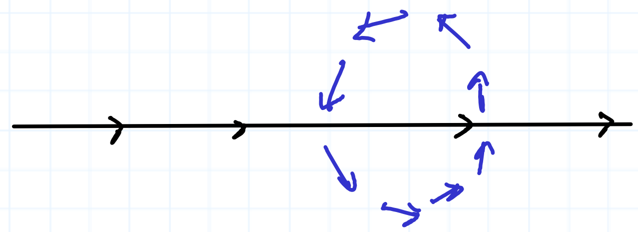
Consider current flowing in a wire:  .

We have $E = 0$ since there is no build-up of charge. By Maxwell,

$\rho = 0$

$$\nabla\times B = \mu_0 J + \underbrace{\mu_0\varepsilon_0 \frac{\partial E}{\partial t}}_{0} = \mu_0 J.$$

We get a circulation of the magnetic field in a plane perpendicular to $\mathbb{J}$:

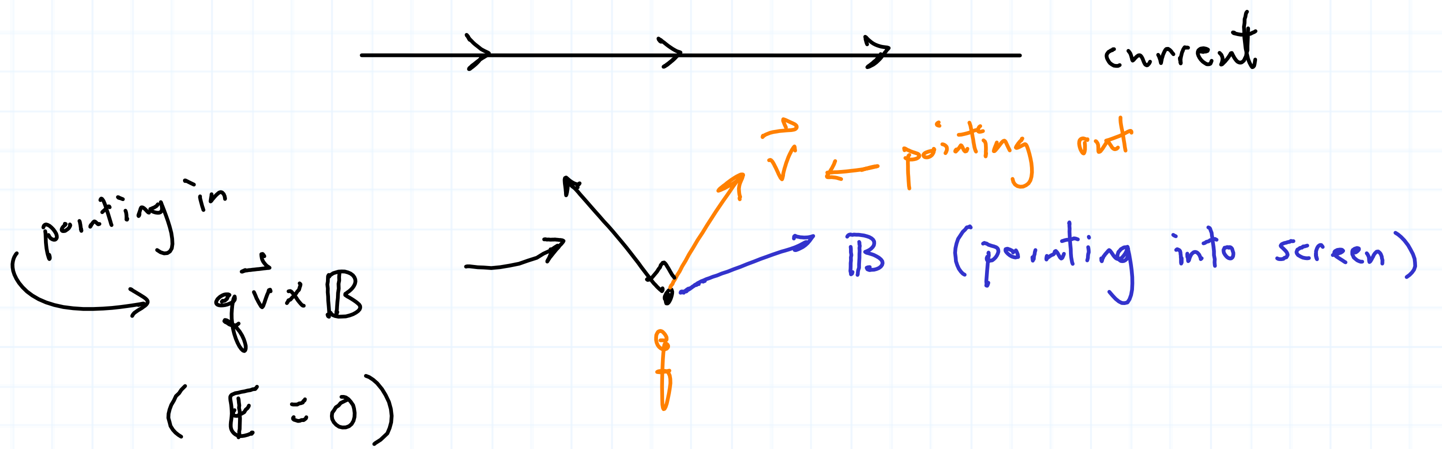


← As observed in nature.

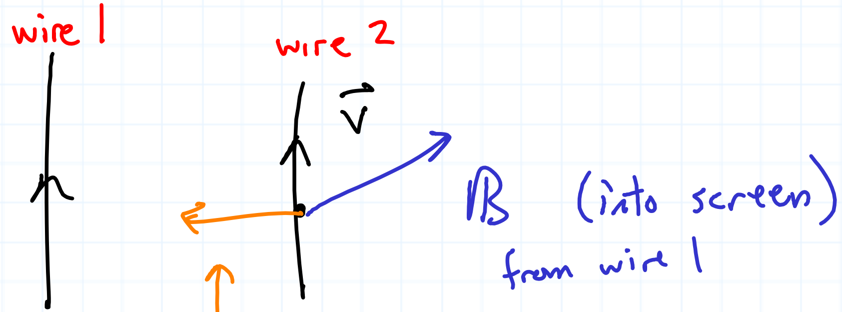
Lorentz Law ← Not derivable from Maxwell's equations

The force on a point charge q moving with velocity $\vec{v}$ in the presence of an electric field $\mathbb{E}$ and magnetic field $\mathbb{B}$ is

$$\mathbb{F} = q\,(\mathbb{E} + \vec{v} \times \mathbb{B})$$



Two wires carrying current exert a force on each other!



$\mathbb{F} = q(v \times \mathbb{B})$ pointing towards other wire

force from wire 1

Note: The force due to a magnetic field does no work on the charge:
$\vec{v}$ is perpendicular to $\vec{v} \times \mathbb{B}$.