# Math 212 Maxwell's equations

$$\nabla \cdot \mathbb{E} = \frac{1}{\varepsilon_0}\rho \quad \text{(Gauss' law)}$$

$$\nabla \cdot \mathbb{B} = 0$$

$$\nabla \times \mathbb{E} = -\frac{\partial \mathbb{B}}{\partial t} \quad \text{(Faraday's law)}$$

$$\nabla \times \mathbb{B} = \mu_0 \mathbb{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbb{E}}{\partial t} \quad \text{(Ampère's law w/ Maxwell's correction)}$$

$\mathbb{E}$ = electric field, $\mathbb{B}$ = magnetic field, $\mathbb{J}$ = current density, $\rho$ = charge density

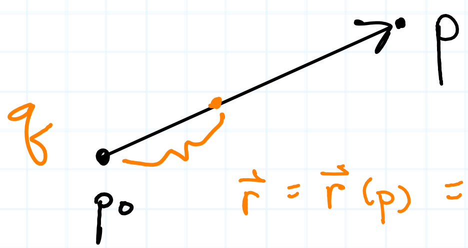
## Point charges

A point charge of $q$ Coulombs located at a point $p_0 = (x_0, y_0, z_0)$ has an electric field

$\vec{r}$ = unit vector in the direction from $p_0$ to $p$, and $r = |p - p_0|$

$$\mathbb{E}_q(p) = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \vec{r} = \frac{1}{4\pi\varepsilon_0} \frac{q}{|p-q|^2} \frac{p-p_0}{|p-p_0|} = \frac{1}{4\pi\varepsilon_0} q \frac{p-p_0}{|p-p_0|^3}$$

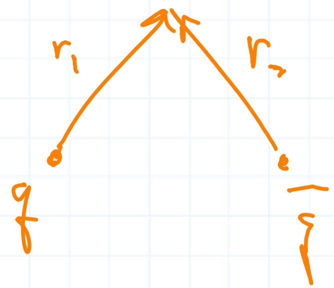
$\varepsilon_0$ = permittivity of free space $\approx 8.85 \times 10^{-12} \frac{C^2}{N \cdot m^2}$



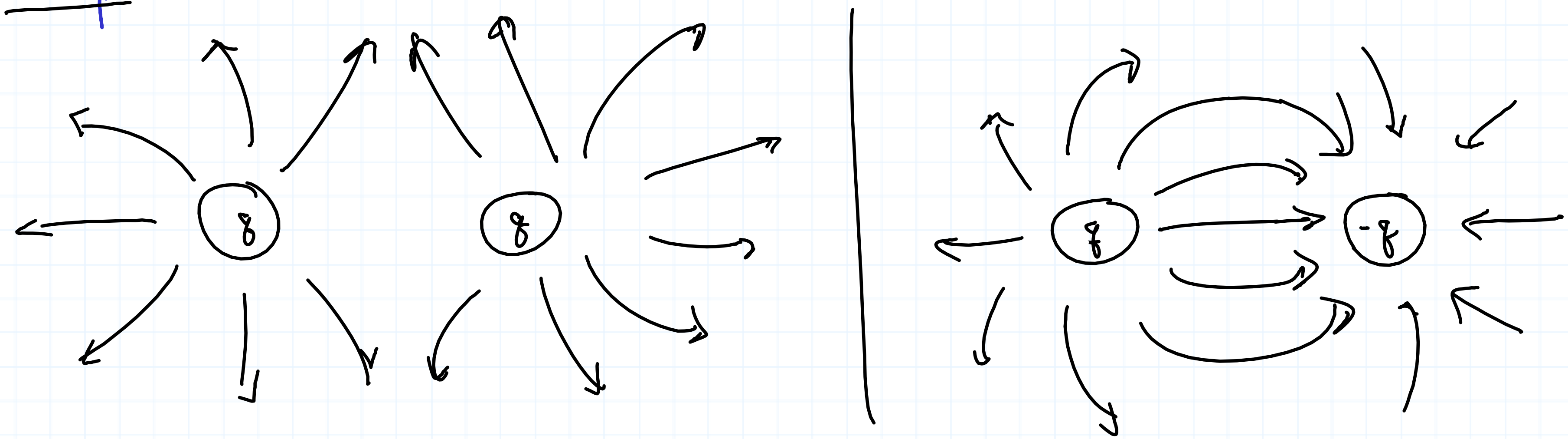

$\vec{r} = \vec{r}(p) = \frac{p - p_0}{r}$, $r = |p - p_0|$

## Superposition rule:

For two charges $q, \tilde{q}$, the electric field is given by $\mathbb{E}_q + \mathbb{E}_{\tilde{q}}$.




Example

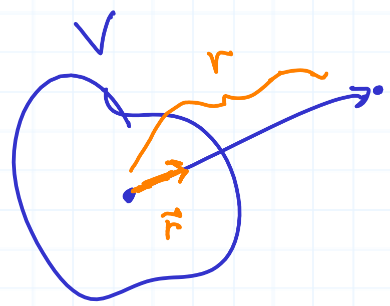



(see Sage)

For charges $q_1, \dots, q_r$, $E = \sum_{i=1}^{r} E_{q_i}$

For a continuous distribution of charge with density $\rho$, distributed in a solid $V$,

$$\mathbb{E} = \frac{1}{4\pi\varepsilon_0}\iiint_V \frac{\vec{r}}{r^2}\,\rho$$



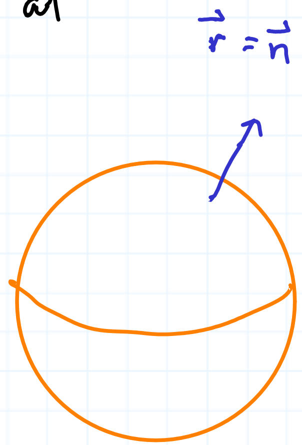

(integrate each component separately:

$$\iiint_V (\alpha,\beta,\gamma) := \left(\iiint\alpha,\ \iiint\beta,\ \iiint\gamma\right)$$ ).

## Flux from a point charge

Let $S$ be a sphere of radius $R$ centered on a charge $q$.

Then $\mathbb{E}$ is a radial vector field and its normal component at the surface of the sphere is




$$\mathbb{E}\cdot n = \frac{1}{4\pi\varepsilon_0}\, q\, \frac{\vec{r}}{R^2}\cdot\vec{n}$$

$$= \frac{1}{4\pi\varepsilon_0}\,\frac{q}{R^2}.$$

($\vec{r} = \vec{n}$, $\vec{r}\cdot\vec{r} = 1$)

Hence, the flux (surface area weighted by the normal component) is

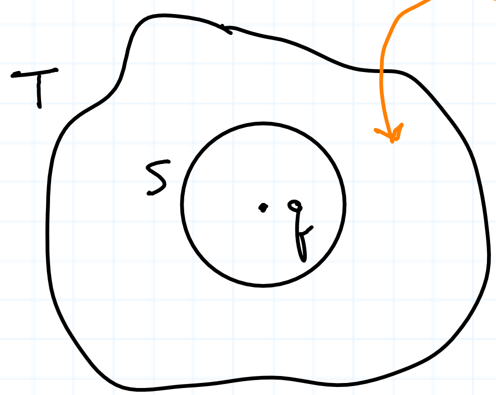
$$\left(\frac{1}{4\pi\varepsilon_0}\frac{q}{R^2}\right)(4\pi R^2) = q/\varepsilon_0 .$$

## Flux through any closed surface containing a point charge

Let $T$ be any closed surface containing $q$. Choose a sphere $S$ centered at $q$ with small enough radius so that $S \subseteq T$.

Exercise If $E = \frac{1}{4\pi\varepsilon_0} q \frac{\vec{r}}{r^2}$, show $\operatorname{div} E = 0$.

div $E = 0$ between $S$ and $T$




Let $V$ denote the volume between $S$ and $T$.

Then $$0 = \int_V \operatorname{div} E \overset{\text{Stokes'}}{=} \int_T E\cdot\vec{n} - \int_S E\cdot\vec{n}$$

$$\Longrightarrow \int_T E\cdot\vec{n} = \int_S E\cdot\vec{n} = q/\varepsilon_0 .$$

So the flux of a point charge $q$ through any surface $T$ is $q/\varepsilon_0$.

Note: What's wrong with the following argument? Let $T$ be a closed surface containing a point charge $q$. Since $\operatorname{div} E = 0$ for a point charge,

$$\text{flux of } E \text{ through } T = \int_T E\cdot\vec{n} \overset{\text{Stokes'}}{=} \int_{\substack{\text{volume}\\ \text{inside } T}} \operatorname{div} E = 0.$$

Answer: $\operatorname{div} E$ is not defined at $0$, so the hypotheses of Stokes' theorem are not met.

## Flux from several point charges

If $S$ is a surface containing point charges $q_1, \dots, q_k$, then $E = \sum_{i=1}^{k} E_{q_i}$ and the flux of $E$ through $S$ is

$$\iint_S E\cdot\vec{n} = \sum_i \iint_S E_i = \frac{1}{\varepsilon_0}\Big(\sum_i q_i\Big) = \left(\begin{matrix}\text{enclosed}\\ \text{charge}\end{matrix}\right)\Big/\varepsilon_0 .$$

The flux through the boundary of a solid $V$ due to a continuous distribution of charge $\rho$ is

Maxwell's equation: $\nabla \cdot E = \rho/\varepsilon_0$.

$$\iint_{\partial V} E \cdot \vec{n} \overset{\text{Stokes'}}{=} \iiint_V \nabla \cdot E = \frac{1}{\varepsilon_0} \underbrace{\iiint_V \rho}_{\text{enclosed charge}} .$$

Note: I have been assuming that all parametrizations of closed surfaces are chosen so that the unit normal points out of the surface. (So "flux" means flux out of the surface.)