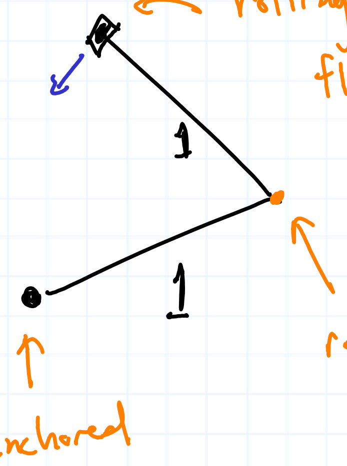
Tomorrow's quiz: see website.

Review HW

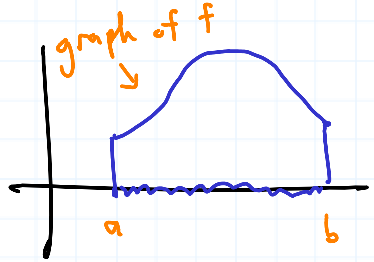
Planimeter



← rolling wheel recording the flow of the unit vector field F in the direction of the blue arrow (perpendicular to the arm).

rotational joint

One may calculate that the vector field F satisfies $\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y} = 1$. Hence, tracing the wheel around a curve will measure the area enclosed. Planimeters can be used to integrate functions



← Trace the blue curve to find the area, $\int_a^b f$.

k-surface area

Let $D \subseteq \mathbb{R}^k$ be a bounded set, and let $S: D \to \mathbb{R}^n$. Define

$$g_{ij} = \frac{\partial S}{\partial x_i} \cdot \frac{\partial S}{\partial x_j}$$

for $i,j \in [1,\dots,k]$, and let $g = (g_{ij})$, a $k \times k$ matrix.

If f is a function defined on the image of S, define the weighted surface area of S by

$$\text{area}_f(S) = \int_S f := \int_D f \circ S \sqrt{\det g} \,.$$

Explanation: Choose a box containing D and partition it. Then

$$\text{area}_f(S) \approx \sum_J f(S(x_J)) \,\text{vol}(S(J)) \approx \sum_J f(S(x_J)) \,\overbrace{\text{vol}_k}^{\text{k-dim'l volume} \leftarrow \text{to be defined}}(B_J)\, \text{vol}(J)$$

subboxes of partition

sample point $x_J \in J$

B_J = box spanned by the columns of $JS(x_J)$, i.e., the $\frac{\partial S}{\partial x_i}(x_J)$.

We now show its reasonable to define the k-dimensional volume of B_J to be $\sqrt{\det g(x_J)}$.

Let $v_1, \dots, v_k \in \mathbb{R}^n$. Let Π any k-dim'l subspace of $\mathbb{R}^n$ containing $v_1, \dots, v_k$ (uniquely determined if $v_1, \dots, v_k$ are linearly independent).

Pick an orthonormal basis $e_1, \dots, e_k$ for Π, i.e., $e_i \cdot e_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$

Write $v_i = \sum_{j=1}^{k} a_{ij} e_j$. Having chosen the basis $\{e_i\}$, we can identify Π with $\mathbb{R}^k$ and each v_i as $(a_{i1}, \dots, a_{ik})$.

So it is reasonable to define the volume of the box spanned by $v_1, \dots, v_k$ by $|\det A|$ where $A = (a_{ij})$, i.e. A is the matrix whose rows are the v_i's in the coordinates for Π.

Now $v_i \cdot v_j = \left(\sum_{s=1}^{k} a_{is} e_s \right) \cdot \left(\sum_{t=1}^{k} a_{jt} e_t \right)$

$$= \sum_{s,t=1}^{k} a_{is} a_{jt} \underbrace{e_s \cdot e_t}_{= 0 \text{ unless } s = t}$$

$$= \sum_{s=1}^{k} a_{is} a_{js}$$

$$= \begin{bmatrix} a_{i1} & \cdots & a_{ik} \end{bmatrix} \begin{bmatrix} a_{j1} \\ \vdots \\ a_{jk} \end{bmatrix}$$

transpose of A

$= ij^{\text{th}}$ entry of $A A^t$.

Let $g = (v_i \cdot v_j)$, a $k \times k$ matrix. Then

$$g = A A^t \Rightarrow \det g = \det(A A) = \det A \det A^t$$

$$= \det A \det A = (\det A)^2.$$

Hence, $|\det A| = \sqrt{\det g}$, as required.

Remark This formula for k-dim'l surface area agrees with the formula for surface area in the special cases we have already considered:

(i) weighted length of a curve in $\mathbb{R}^n$: $\int_a^b f \circ c \, |c'|$

In this case, g is the 1×1 matrix with single entry $c' \cdot c' = \sum c_i'(t)^2$. So $\sqrt{|\det g|} = |c'|$.

(ii) weighted surface area for $S : D \to \mathbb{R}^3$, $D \subseteq \mathbb{R}^2$:

$$\int_D f \circ S \, |S_u \times S_v|.$$

Here, g is the 2×2 matrix $\begin{bmatrix} S_u \cdot S_u & S_u \cdot S_v \\ S_v \cdot S_u & S_v \cdot S_v \end{bmatrix}$. Calculate: $\sqrt{\det g} = |S_u \times S_v|$.

Example The surface area of $S: [0,1]^2 \longrightarrow \mathbb{R}^4$

$$(u,v) \longmapsto (u^3, u^2v, uv^2, v^3)$$

is $\int_{[0,1]^2} \sqrt{\det g}$ where

$$S_u = (3u^2, 2uv, v^2, 0)$$

$$S_v = (0, u^2, 2uv, 3v^2)$$

$$g = \begin{bmatrix} S_u \cdot S_u & S_u \cdot S_v \\ S_v \cdot S_u & S_v \cdot S_v \end{bmatrix}$$

$$= \begin{bmatrix} 9u^4 + 4u^2v^2 + v^4 & 2u^3v + 2uv^3 \\ 2u^3v + 2uv^3 & u^4 + 4u^2v^2 + 9v^4 \end{bmatrix}.$$

Yuck.