# Math 212

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Last time $V: D \to \mathbb{R}^3$ ← solid in $\mathbb{R}^3$

$D \subset \mathbb{R}^3$

$F: \mathbb{R}^3 \to \mathbb{R}^3$ ← vector field in $\mathbb{R}^3$

flux form

flux of F through $\partial V$: $\int_{\partial V} F \cdot \vec{n} = \int_{\partial V} \omega^F$

Stokes'

$$= \int_V d\omega^F$$

$$= \int_V (\underbrace{D_1F_1 + D_2F_2 + D_3F_3}_{\text{div } F})\, dx \wedge dy \wedge dz$$

$$= \int_V \nabla \cdot F \, dx \wedge dy \wedge dz$$

def. of integration of a diff'l form

$$= \int_D (\nabla \cdot F) \circ V \, \det JV$$

$$= \int_D (\nabla \cdot F) \circ V \, |\det JV| \quad \text{if } \det JV \geq 0$$

def. of weighted volume of V

$$= \int_V \nabla \cdot F$$

**Def.** The divergence of $F$ is $\text{div } F := \nabla \cdot F$

$$= \sum_{i=1}^{3} D_i F_i$$

Example $F(x,y,z) = (x,y,z)$, $V$ = parametrization of solid ball of radius $R$ centered at the origin

$$S = \partial V.$$

The flux of $F$ through $S$ = surface area of $S$ weighted by the component of $F$ in the normal direction,

$$\vec{n} = \frac{(x,y,z)}{\sqrt{x^2+y^2+z^2}} = \frac{1}{R}(x,y,z)$$

$$= (4\pi R^2)\left[F \cdot \tfrac{1}{R}(x,y,z)\right] = 4\pi R^2\left[(x,y,z)\cdot \tfrac{1}{R}(x,y,z)\right]$$

$$= 4\pi R^2\left[\frac{x^2+y^2+z^2}{R}\right] = 4\pi R^2\left[\frac{R^2}{R}\right] = 4\pi R^3$$

On the other hand, we can calculate this by

$$\int_V \operatorname{div} F = \int_V 3 = 3\left(\tfrac{4}{3}\pi R^3\right) = 4\pi R^3.$$

Note! It's easy to see from the second calculation that being centered at the origin is not important.

Claim: The divergence of $F$ measures flux per unit volume.

Given $p \in \mathbb{R}^3$, let $B_\varepsilon$ be a (parametrization of a) ball of radius $\varepsilon$ centered at $p$.

Then the flux of $F$ through $\partial B_\varepsilon$ is

$$\iint_{\partial B_\varepsilon} F \cdot \vec{n} \overset{\text{Stokes'}}{=} \iiint_{B_\varepsilon} \operatorname{div} F$$

$$= \text{volume of } B_\varepsilon \text{ weighted by } \operatorname{div} F$$

$$\approx \operatorname{div} F(p) \operatorname{vol}(B_\varepsilon)$$ (assuming $\operatorname{div} F$ does not vary much on $B_\varepsilon$)

$$\Longrightarrow \operatorname{div} F(p) \approx \frac{1}{\operatorname{vol}(B_\varepsilon)} \iint_{\partial B_\varepsilon} F \cdot \vec{n} \xrightarrow[\text{as } \varepsilon \to 0]{} \text{"flux density" at } p.$$

**Theorem.** The following are equivalent for a vector field $F: \mathbb{R}^3 \to \mathbb{R}^3$

1) $\nabla \cdot F = 0$ $\quad (\operatorname{div} F = 0)$.
2) $\iint_S F \cdot \vec{n}$ is independent of $S$ as long as the boundary is not changed.
3) $\oiint_S F \cdot \vec{n} = 0$ if $S$ is a closed surface (i.e., $\partial S = \emptyset$).
4) $F = \nabla \times G$ $\quad (F = \operatorname{curl} G)$ for some vector field $G$.

In this case $G$ is called a vector potential for $F$.

Easy parts of proof / $1 \Rightarrow 3$ and $4 \Rightarrow 2$ by Stokes' thm.

$4 \Rightarrow 1$ since $d^2 = 0$ ( $d\omega_G = \omega^{\operatorname{curl} G} = \omega^F$ if $F = \operatorname{curl} G$. Then

$$0 = d^2 \omega_G = d\omega^F = \operatorname{div} F \, dx \wedge dy \wedge dz \ ).$$

Compare this with our earlier result, one dimension down:

**Thm.** The following are equivalent for a vector field $F: \mathbb{R}^3 \to \mathbb{R}^3$.

1) $\nabla \times F = 0$ (curl $F = 0$)
2) $\int_C F \cdot \vec{t}$ is independent of as long as the endpoints stay the same.
3) $\oint_C F \cdot t = 0$ if $C$ is a closed curve.
4) $F = \nabla \varphi$ for some function $\varphi$ (so $F$ has a potential & $F = \text{grad}\, \varphi$).

**Note:** Both part 4s are dependent on $F$ being defin on all of $\mathbb{R}^3$.
For example, in our HW last week, we considered

$$\omega = \frac{y\,dx - x\,dy}{x^2 + y^2}$$ and saw $d\omega = 0$, yet, $\omega \neq d\varphi$ for any function $\varphi$. (So $F = \left(\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right)$ has curl $F = 0$, but $F$ has no potential.)