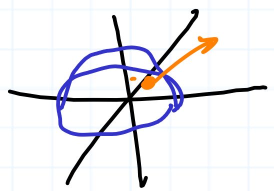
Math 212

Last time $S : D \to \mathbb{R}^3$, $F : \mathbb{R}^3 \to \mathbb{R}^3$
$D \subset \mathbb{R}^2$

Stokes' $\oint_{\partial S} F \cdot \vec{t} = \int_S \operatorname{curl} F \cdot \vec{n}$.

Note: Stokes' implies $\int_S \operatorname{curl} F \cdot \vec{n}$ only depends on ∂S.

Example Let S be a (parametrization of a) sphere of radius 1 in $\mathbb{R}^3$ centered at the origin, and let $F = (x-y, 3z^2, -x)$. Find the flux of curl F through S. (Suppose the parametrization is such that the normal to the surface has non-negative z-component always:



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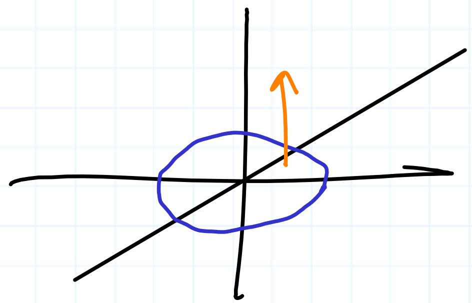
Solution/ By Stokes' we can replace S by any surface with the same boundary, say the flat disk of radius 1 in the xy-plane.



We have $\operatorname{curl} F = \nabla \times F = \begin{vmatrix} i & j & k \\ D_1 & D_2 & D_3 \\ x-y & 3z^2 & -x \end{vmatrix}$

$= (-6z, -1, 1)$. On the disk, $z = 0$, so the curl there is $(0,-1,1)$. The unit normal is $(0,0,1)$. The component of curl F in the normal direction is $(0,-1,1) \cdot (0,0,1) = 1$. So the area of the disk weighted by this component is just the normal area of the disk, i.e. π. Therefore, $\int_S \operatorname{curl} F \cdot \vec{n} = \pi$.

Check via Stokes': Parametrize ∂S by $c(t) = (\cos t, \sin t, 0)$, $0 \le t \le 2\pi$.

Then $\int_{\partial S} F \cdot t = \int_0^{2\pi} (F \circ c) \cdot c' = \int_0^{2\pi} (\cos t - \sin t, 0, -\cos t) \cdot (-\sin t, \cos t, 0)$

$$= \int_0^{2\pi} -\cos t \sin t + \sin^2 t = -\frac{\sin^2 t}{2}\Big|_0^{2\pi} + \int_0^{2\pi} \overbrace{\frac{1-\cos 2t}{2}}^{\sin^2 t} = \frac{1}{2}\int_0^{2\pi} 1 - \cos 2t = \frac{1}{2}\left[t - \frac{\sin 2t}{2}\right]_0^{2\pi}$$

$= \pi$. ✓

Next classical case [after $n=1$ (curves) and $n=2$ (surfaces)]

note these n's are the same. The main case is $n=3$.

Def. Let $V: D \to \mathbb{R}^n$ with $D \subseteq \mathbb{R}^n$ and assume V is continuously differentiable. Let $f: \mathbb{R}^n \to \mathbb{R}^n$ be continuous. The weighted volume of V is

$$\int_V f := \int_D f \circ V \, |\det JV| .$$

Remark: This is just our earlier change of variables theorem provided V is injective almost everywhere.

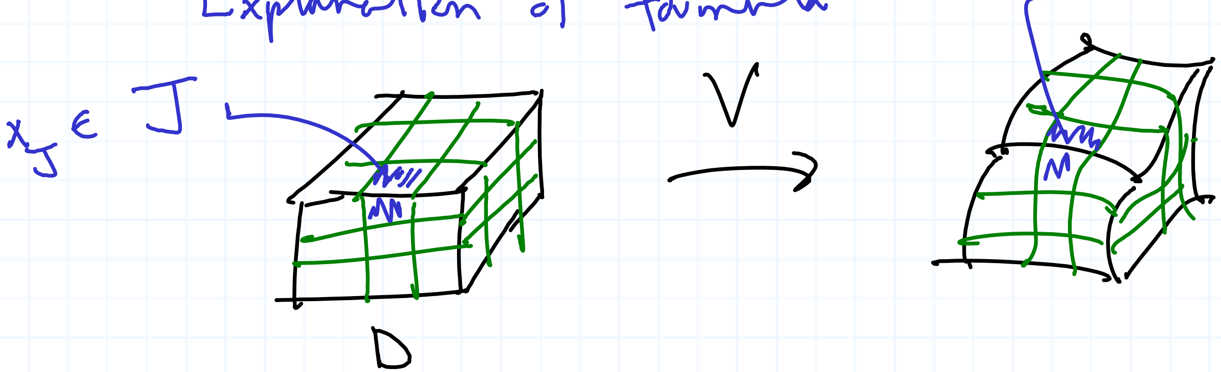
In particular, $\mathrm{vol}(V) = \int_V 1 = \int_D |\det JV|$.

Example $V = V(r, \theta, \varphi)$, spherical coordinates, $0 \le r \le R$, $0 \le \theta \le 2\pi$, $0 \le \varphi \le \frac{\pi}{2}$.

Then $\mathrm{vol}(V) = \frac{4}{3}\pi r^3$.

$V(J) \ni V(x_J)$

Explanation of formula



f-weighted volume of V

$$\approx \sum_J f(V(x_J))\,\mathrm{vol}(V(J))$$

unsigned stretching factor

$$= \sum_J f(V(x_J))\,|\det JV(x_J)|\,\mathrm{vol}(J)$$

$$\approx \int_D f \circ V \, |\det JV|$$

Remark: We have already defined the volume of the <u>set</u> $V(D) \subseteq \mathbb{R}^n$. The volume of (the function V) we have just defined will give the volume of $V(D)$ provided 1 V is injective almost everywhere and 2 $\det JV$ does not change sign (changing orientation).

Stokes: $V : D \to \mathbb{R}^3$, $F : \mathbb{R}^3 \to \mathbb{R}^3$
$D \subseteq \mathbb{R}^3$

flux form for F: $\omega^F = F_1\, dy \wedge dz - F_2\, dx \wedge dz + F_3\, dx \wedge dy$.

The flux of F through the boundary of V is

$$\int_{\partial V} \omega^F \overset{\text{Stokes'}}{=} \int_V d\omega^F.$$

Calculate: $d\omega^F = (D_1 F_1\, dx + D_2 F_1\, dy + D_3 F_1\, dz) \wedge dy \wedge dz$ − blah + blah

$$= D_1 F_1\, dx \wedge dy \wedge dz + D_2 F_2\, dy \wedge dx \wedge dz + D_3 F_3\, dz \wedge dx \wedge dy$$

$$= (D_1 F_1 + D_2 F_2 + D_3 F_3)\, dx \wedge dy \wedge dz.$$

Def. The divergence of F is $\operatorname{div} F = D_1 F_1 + D_2 F_2 + D_3 F_3 = \nabla \bullet F$.

(Note: $\operatorname{div} F : \mathbb{R}^3 \to \mathbb{R}$.)

<u>Stokes</u> flux of F through $\partial V = \int_{\partial V} F \cdot \vec{n}$

$$= \int_{\partial V} w^F$$

$$\overset{\text{Stokes}}{=} \int_V dw^F$$

$$= \int_V \operatorname{div} F \, dx \wedge dy \wedge dz$$

← pulling back involve det JV, without absolute values.

$$= \int_V \operatorname{div} F \qquad (\text{provided } \det JV \geq 0)$$

Sometimes stated as: $\iiint_V \operatorname{div} F = \oiint_S F \cdot \vec{n}$ where $S = \partial V$.