Math 212

Quiz

1. $C(t) = (\sin t - t\cos t, \cos t + t\sin t, t^2)$, $0 \le t \le 1$. Find the length of C.

2. Find the flow of $F(x,y) = (-y, x)$ along $C(t) = (t, t^2)$ for $0 \le t \le 1$.

3. Explain why Stokes' theorem implies that the flow of a gradient vector field F along a curve C is given by the change in potential.

Last time $F: \mathbb{R}^3 \to \mathbb{R}^3$, $\operatorname{curl} F = \nabla \times F$ where $\nabla = (D_1, D_2, D_3)$

If $S: D \to \mathbb{R}^3$, then $\int_{\partial S} F \cdot \vec{t} = \int_S \operatorname{curl} F \cdot \vec{n}$.

$D \subseteq \mathbb{R}^2$

Example $S(u,v) = (u, v, uv)$, $D = [0,1]^2$, $F = (x, xy^2, y+z)$

∂S: $\quad C_1(t) = S(0,t) = (0,t,0)$, $\quad C_2(t) = S(1,t) = (1,t,t)$

$C_3(t) = S(t,0) = (t,0,0)$, $\quad C_4(t) = S(t,1) = (t,1,t)$

Then $\partial S = -C_1 + C_2 + C_3 - C_4$.

$+$ since $(-1)^{i+\alpha} = 1$ when $i=2$, $\alpha = 0$ (setting second variable equal to 0)

So $\displaystyle\oint_{\partial S} F = -\int_{C_1} F + \int_{C_2} F + \int_{C_3} F - \int_{C_4} F$

$$= -\int_0^1 (F\circ C_1)\cdot C_1' + \int_0^1 (F\circ C_2)\cdot C_2' + \int_0^1 (F\circ C_3)\cdot C_3' - \int_0^1 (F\circ C_4)\cdot C_4'$$

$$= \int_0^1 -F(0,t,0)\cdot(0,1,0) + F(1,t,t)\cdot(0,1,1) + F(t,0,0)\cdot(1,0,0) - F(t,1,t)\cdot(1,0,1)$$

$$= \int_0^1 -(0,0,t)\cdot(0,1,0) + (1,t^2,2t)\cdot(0,1,1) + (t,0,0)\cdot(1,0,0) - (t,t,1+t)\cdot(1,0,1)$$

$$= \int_0^1 -0 + t^2 + 2t + t - t - (1+t) = \int_0^1 t^2 + t - 1 = \frac{2}{3} + \frac{1}{2} - 1 = -\frac{1}{6}.$$

This gives the circulation of F about the boundary of S.
We now check that this is the same as the flux of the curl of F through S.

$$\text{curl}\, F = \nabla \times F = \begin{vmatrix} i & j & k \\ D_1 & D_2 & D_3 \\ x & xy^2 & y+z \end{vmatrix} = (1, 0, y^2)$$

$$S_u \times S_v = \begin{vmatrix} i & j & k \\ 1 & 0 & v \\ 0 & 1 & u \end{vmatrix} = (-v, -u, 1)$$

$(\text{curl} F)(u, v, uv) = (1, 0, v^2)$

$$\int_S \text{curl} F \cdot \vec{n} = \int_D [(\text{curl} F) \circ S] \cdot (S_u \times S_v)$$

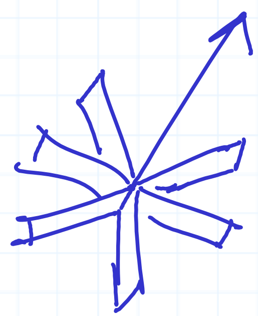
$$= \int_D (1, 0, v^2) \cdot (-v, -u, 1)$$

$$= \int_0^1 \int_0^1 -v + v^2 \, du \, dv = \int_0^1 -v + v^2 \, dv$$

$$= -\frac{1}{2} + \frac{1}{3} = -\frac{1}{6}. \checkmark$$

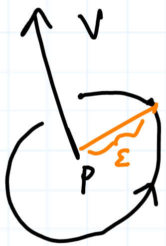
Geometric Meaning of the Curl

Claim: Let v be a unit vector in $\mathbb{R}^3$, and let $p \in \mathbb{R}^3$ be any point. Then the component of curl F in the direction of v is the "circulation density" of F in the plane passing through p and perpendicular to v. So curl F points in the direction of maximal circulation density. Imagine an arrow with a base consisting of paddles:



Suppose that F pushes against these paddles, causing a spin. Rotating the base of the contraption at the point p, it will spin quickest when the arrow is pointing in the direction of curl F.

Explanation:



Let D_ε be the disc of radius ε centered at p and perpendicular to v.

Then the flux of curl F through $D_\varepsilon = \int_{D_\varepsilon} \text{curl} F \cdot \vec{n}$ = surface area of D_ε weighted by the component of curl F perpendicular to D_ε.

$$\approx [\text{curl} F(p) \cdot v] \text{ area } D_\varepsilon$$

By Stokes' $\int_{D_\varepsilon} \text{curl} F \cdot \vec{n} = \underbrace{\oint_C F \cdot \vec{t}}_{\text{circulation of } F \text{ about } \partial D_\varepsilon}$

where $C = \partial D_\varepsilon$, the circle of radius ε. Thus,

$$[\text{curl} F(p) \cdot v] \text{ area } D_\varepsilon \approx \oint_C F \cdot \vec{t} \implies \underbrace{\frac{1}{\text{area } D_\varepsilon} \oint_C F \cdot \vec{t}}_{\text{circulation per unit area}} = \underbrace{\text{curl} F(p) \cdot v}_{\text{component of curl} F}.$$