Math 212

(recall: weighted surface area $\int_S f = \int_D f\circ S\,|S_u \times S_v|$)

Last time $S : D \to \mathbb{R}^3$ surface, $F : \mathbb{R}^3 \to \mathbb{R}^3$ vector field

$$\omega^F = F_1\, dy\wedge dz - F_2\, dx \wedge dz + F_3\, dy\wedge dz$$ flux form

$\left(\begin{array}{l} \omega^F = *\,\omega_F \\ *\,\omega^F = \omega_F \end{array} \right)$

"Hodge star" operator

$$\int_S F\cdot d\vec{n} := \int_S \omega_F$$ flux of F through S

Explanation $S = S(u,v) = (S_1, S_2, S_3)$ w/ $S_i : D \to \mathbb{R}$

$$\int_S \omega^F = \int_D S^*\omega^F = \int_D F_1\circ S\; dS_2\wedge dS_3 - F_2\circ S\; dS_1 \wedge dS_3 + F_3\circ S\; dS_2\wedge dS_3$$

$$= \int_D F_1\circ S \left(\frac{\partial S_2}{\partial u}du + \frac{\partial S_2}{\partial v}dv\right)\wedge\left(\frac{\partial S_3}{\partial u}du + \frac{\partial S_3}{\partial v}dv\right) - F_2\circ S\,(\quad)\wedge(\quad) + F_3\circ S\,(\quad)\wedge(\quad)$$

$= $ etc.

Easier way to calculate this: Let $S_{iu} = \dfrac{\partial S_i}{\partial u}$, $S_{iv} = \dfrac{\partial S_i}{\partial v}$ $\quad i = 1, 2, 3.$

$$JS = \begin{array}{cc} \begin{bmatrix} S_{1u} & S_{1v} \\ S_{2u} & S_{2v} \\ S_{3u} & S_{3v} \end{bmatrix} & \begin{matrix} dx \\ dy \\ dz \end{matrix} \\ \begin{matrix} \uparrow \\ \frac{\partial S}{\partial u} \end{matrix} \quad \begin{matrix} \uparrow \\ \frac{\partial S}{\partial v} \end{matrix} & \end{array}$$

$| \; | = \det[\;]$

$$\int_S \omega^F = \int_D F_1 \circ S \begin{vmatrix} S_{2u} & S_{2v} \\ S_{3u} & S_{3v} \end{vmatrix} - F_2 \circ S \begin{vmatrix} S_{1u} & S_{1v} \\ S_{3u} & S_{3v} \end{vmatrix} + F_3 \circ S \begin{vmatrix} S_{1u} & S_{1v} \\ S_{2u} & S_{2v} \end{vmatrix}$$

dot product

$$= \int_D \underbrace{(F_1 \circ S,\ F_2 \circ S,\ F_3 \circ S)}\ \cdot\ \underbrace{\left(\begin{vmatrix} S_{2u} & S_{2v} \\ S_{3u} & S_{3v} \end{vmatrix}, - \begin{vmatrix} S_{1u} & S_{1v} \\ S_{3u} & S_{3v} \end{vmatrix}, \begin{vmatrix} S_{1u} & S_{1v} \\ S_{2u} & S_{2v} \end{vmatrix} \right)}$$

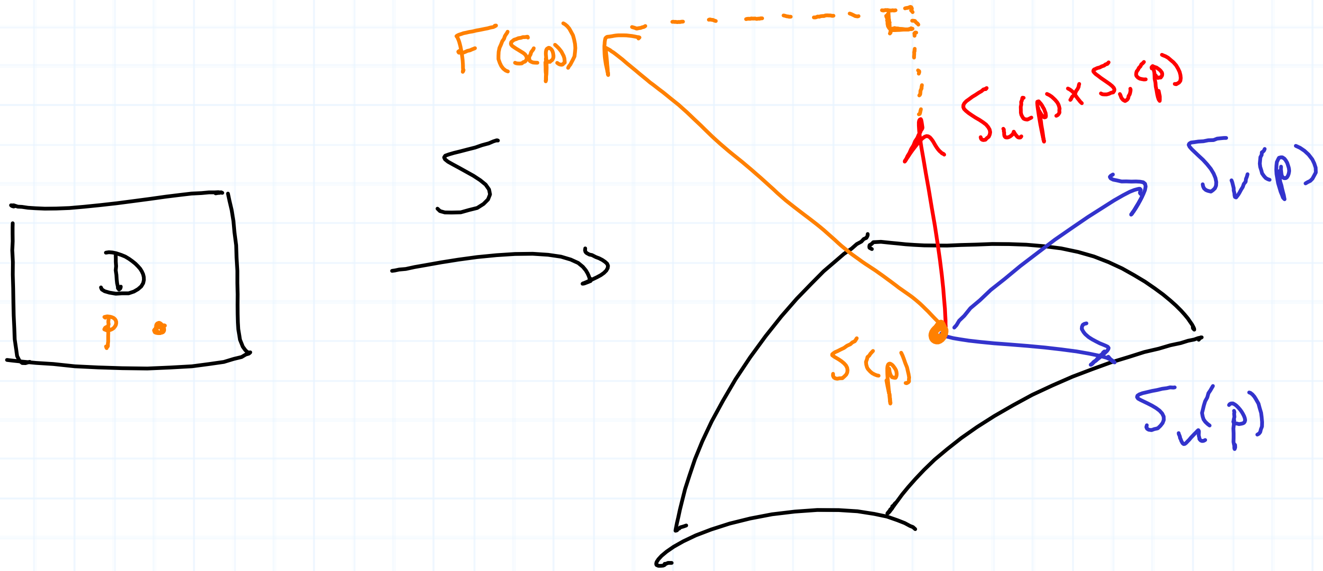
remember ☆☆

$$= \int_D (F \circ S) \cdot (S_u \times S_v)$$

$$S_u \times S_v = \begin{vmatrix} i & j & k \\ S_{1u} & S_{2u} & S_{3u} \\ S_{1v} & S_{2v} & S_{3v} \end{vmatrix}$$

$$= \int_D \left((F \circ S) \cdot \underbrace{\frac{S_u \times S_v}{|S_u \times S_v|}}_{\text{unit normal to } S,\ \vec{n}} \right) \underbrace{|S_u \times S_v|}_{\text{stretching factor for area}}$$

= surface area of S weighted by $(F \circ S) \cdot \vec{n}$,
i.e. weighted by the component of F in the direction normal (i.e., perpendicular) to S.



← $S_u(p)$ + $S_v(p)$ span the tangent space to S at p.

$\boxed{\text{Useful formula}}$ The flux of F through S is

$$\int_S \omega^F = \int_D (F \circ S) \cdot (S_u \times S_v).$$

Example $S(u,v) = (\overset{x}{u}, \overset{y}{v}, \overset{z}{uv})$, $D = [0,1]^2$

$F(x,y,z) = (x, yz, z)$

Calculate the flux of F through S in two ways: $\int_S \omega^F$, $\int_D (F\circ S)(S_u \times S_v)$

I. $\omega^F = x\, dy \wedge dz - yz\, dx \wedge dz + z\, dx \wedge dy$

$$JS = \begin{array}{cc} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ v & u \end{bmatrix} & \begin{array}{c} dx \\ dy \\ dz \end{array} \end{array}$$

$$\int_S \omega^F = \int_D u \begin{vmatrix} 0 & 1 \\ v & u \end{vmatrix} - v(uv) \begin{vmatrix} 1 & 0 \\ v & u \end{vmatrix} + uv \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= \int_D -uv - u^2v^2 + uv = -\int_0^1 \int_0^1 u^2 v^2\, du\, dv = -\int_0^1 \tfrac{1}{3} v^2\, dv = -\frac{1}{9}.$$

II.

$$S_u \times S_v = \begin{vmatrix} i & j & k \\ 1 & 0 & v \\ 0 & 1 & u \end{vmatrix} = (-v, -u, 1)$$

$$\int_S \omega^F = \int_D (F \circ S) \cdot (S_u \times S_v)$$

$F(x,y,z) = (x, yz, z)$

$$= \int_D F(u,v,uv) \cdot (-v,-u,1)$$

$$= \int_D (u, uv^2, uv) \cdot (-v,-u,1)$$

$$= \int_D -uv - u^2v^2 + uv = -\int_D uv = -\frac{1}{9}.$$

(See Sage for picture.)

$D \subseteq \mathbb{R}^{n-1}$

Higher dimensions $\quad S : D \to \mathbb{R}^n$ hypersurface, $F : \mathbb{R}^n \to \mathbb{R}^n$ vector field

$$\omega^F = \sum_{i=1}^n (-1)^{i+1} F_i \, dx_1 \wedge \cdots \wedge \overline{dx_i} \wedge \cdots \wedge dx_n$$ flux form

$$\int_S \omega^F = \int_D (F \circ S) \cdot (S_{u_1} \times \cdots \times S_{u_{n-1}})$$

$$\begin{vmatrix} e_1 & \cdots & e_n \\ S_{u_1,1} & \cdots & S_{u_1,n} \\ \vdots & & \\ S_{u_{n-1},1} & \cdots & S_{u_{n-1},n} \end{vmatrix}$$

Stokes'

$S : D^{\subseteq \mathbb{R}^2} \longrightarrow \mathbb{R}^3, \quad F : \mathbb{R}^3 \to \mathbb{R}^3$

flow form: $\omega_F = F_1\,dx + F_2\,dy + F_3\,dz \in \Omega^1 \mathbb{R}^3$

1-chain: ∂S

flow of F along the boundary $= \int_{\partial S} \omega \overset{\text{Stokes'}}{=} \int_S d\omega_F$

What does $\int_S d\omega_F$ mean?

$D_1 := \frac{\partial}{\partial x}$

$D_2 := \frac{\partial}{\partial y}$

$D_3 := \frac{\partial}{\partial z}$

$$
\begin{aligned}
d\omega_F &= d(F_1\,dx + F_2\,dy + F_3\,dz) \\
&= (D_1F_1\,dx + D_2F_1\,dy + D_3F_1\,dz) \wedge dx \\
&+ (D_1F_2\,dx + D_2F_2\,dy + D_3F_2\,dz) \wedge dy \\
&+ (D_1F_3\,dx + D_2F_3\,dy + D_3F_3\,dz) \wedge dz
\end{aligned}
$$

$$\Rightarrow d\omega_F = (D_2F_3 - D_3F_2)\, dy \wedge dz - (D_3F_1 - D_1F_3)\, dx \wedge dz + (D_1F_2 - D_2F_1)\, dx \wedge dy$$

$=$ flux form for the vector field

$$(D_2F_3 - D_3F_2,\ D_1F_3 - D_3F_1,\ D_1F_2 - D_2F_1)$$