For $v, w \in \mathbb{R}^3$,

$$v \times w = \det \begin{vmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = (v_2 w_3 - w_2 v_3,\ v_3 w_1 - v_1 w_3,\ v_1 w_2 - v_2 w_1).$$

Properties of the cross product

(i) $v \times w$ is perpendicular to the linear span of v and w

(ii) the direction of $v \times w$ is determined by the right-hand rule

(iii) $|v \times w|$ is the area of the parallelogram spanned by v and w:

$$\{av + bw : a \geq 0,\ b \geq 0,\ a + b \leq 1\}.$$

Surface integral $\quad S : D \to \mathbb{R}^3$, $\quad f : \mathbb{R}^3 \to \mathbb{R}$

$D \subseteq \mathbb{R}^2$ $\quad$ surface $\quad$ weighting function

weighted surface area: $\int_S f = \int_S f \, dS := \int_D f \circ S \, |S_u \times S_v|$

Example $S : [0,1]^2 \longrightarrow \mathbb{R}^3$

$(u,v) \longmapsto (u,v,uv)$

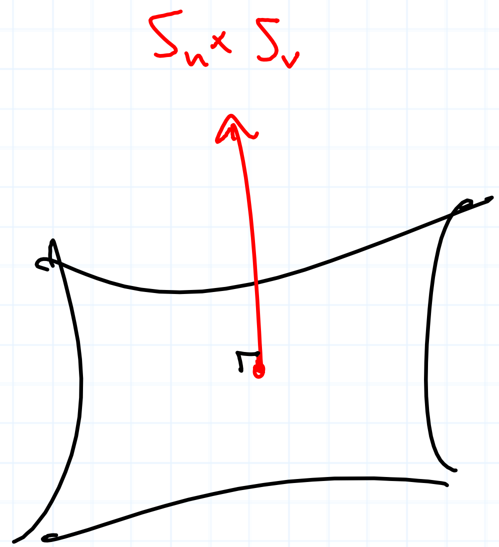
Find area(S).

Solution/ $S_u = (1, 0, v)$, $S_v = (0, 1, u)$

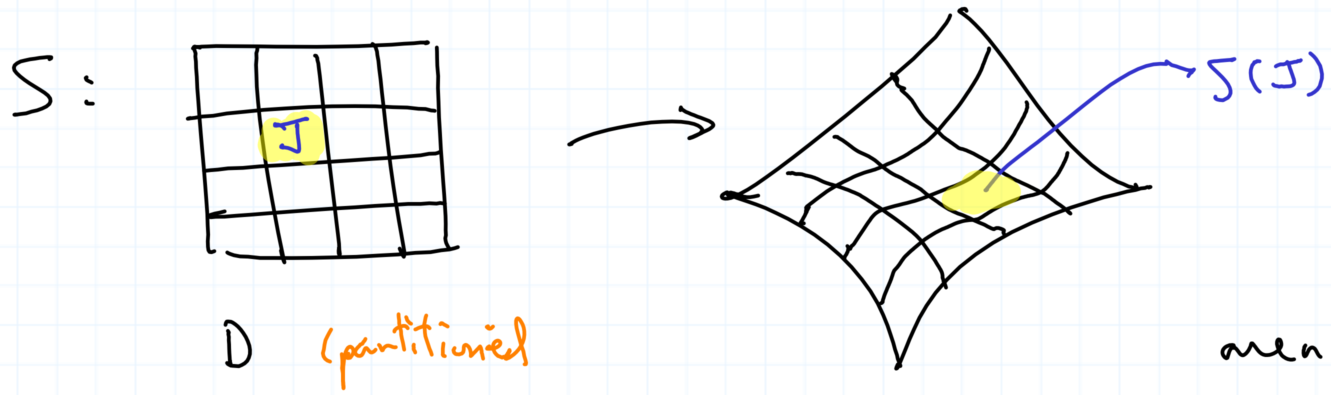
$$S_u \times S_v = \begin{vmatrix} i & j & k \\ 1 & 0 & v \\ 0 & 1 & u \end{vmatrix} = (-v, -u, 1)$$

So $\text{area}(S) = \int_{[0,1]^2} |S_u \times S_v| = \int_{[0,1]^2} \sqrt{1+u^2+v^2}$

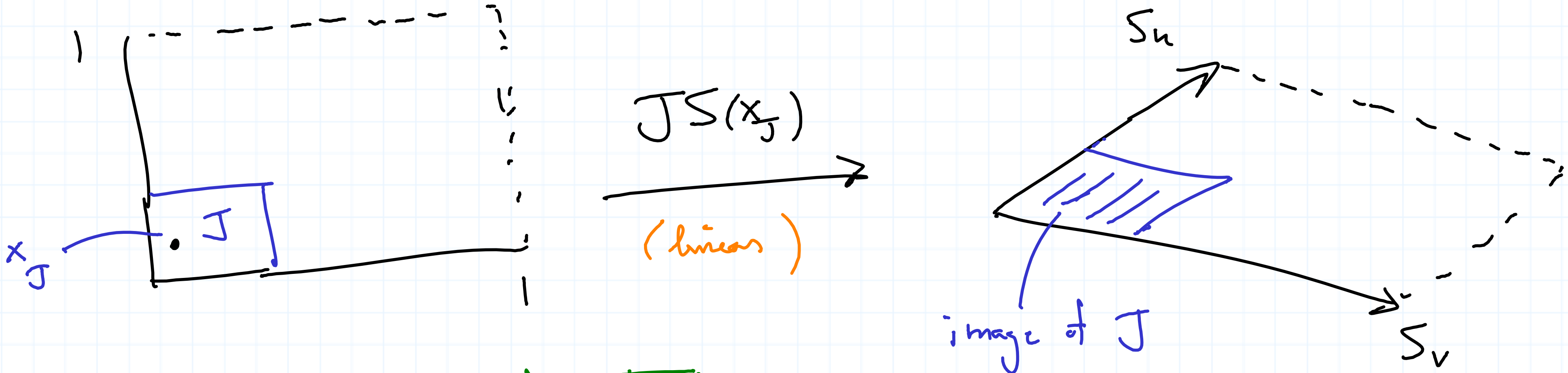
$= \int_0^1 \int_0^1 \sqrt{1+u^2+v^2}\, du\,dv$ = yuck.



Justification for surface area formula



$$\text{area}(S) = \sum \text{area}(S(J))$$



unit area $\longrightarrow$ area of ▱

$$1 \longmapsto |S_u \times S_v|$$

$$\text{vol}(J) \longmapsto |S_u \times S_v|\,\text{vol}(J) = \text{area } JS(x_J)(J) \approx \text{area } S(J)$$

$$\text{weighted area of } S \approx \sum_J f(S(x_J))\,\text{area}(S(J))$$

$$\approx \sum_J f(S(x_J))\,|S_u \times S_v|\,\mathrm{vol}(J)$$

$$\approx \int_D f \circ S\,|S_u \times S_v|\,.$$

$F(x,y,z) = (F_1(x,y,z), F_2(x,y,z), F_3(x,y,z))$

Flux

surface: $S : D \to \mathbb{R}^3$ vector field $F : \mathbb{R}^3 \to \mathbb{R}^3$

Def. flux form $\omega^F = F_1\, dy \wedge dz + F_2\, dz \wedge dx + F_3\, dx \wedge dy$

$\quad = F_1\, dx \wedge dy - F_2\, dx \wedge dz + F_3\, dx \wedge dy$

flux of F through S: $\int_S \omega^F$

(Notation: $\int_S F \cdot d\vec{n}$ or $\int_S F \cdot dS$ where $\vec{n}$ is supposed to remind you of the unit normal to S, $\vec{n} = \dfrac{S_u \times S_v}{|S_u \times S_v|}$.)

Example

(a) $S(u,v) = (u, v, u^2+v^2)$, $F(x,y,z) = (0,0,z)$

$(u,v) \in [0,1]^2 = D$

$$\text{flux} = \int_S \omega^F = \int_S 0\, dy\wedge dz + 0\, dz\wedge dx + z\, dx\wedge dy$$

$$= \int_S z\, dx\wedge dy = \int_D (u^2+v^2)\, du\wedge dv$$

$$= \int_0^1\int_0^1 u^2+v^2\, du\, dv = \frac{2}{3}.$$ ← why $+\frac{2}{3}$?

think →

$(1,0,2u)$

$S_u \times S_v = (1,0,2u)\times(0,1,2v)$

$= (-2u, -2v, 1)$

$S_u\times S_v$ points into the paraboloid, not the other way.

(b) $S(u,v) = (u, v, uv)$, $F(x,y,z) = (x, yz, z)$

$$\text{flux} = \int_S \omega^F = \int_S x\, dy\wedge dz + yz\, dz\wedge dx + z\, dx\wedge dy$$

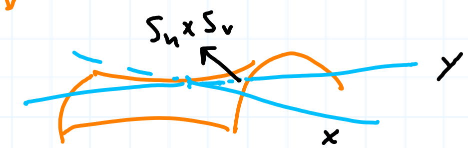
J

$$JS = \begin{array}{cc} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ v & u \end{bmatrix} & \begin{array}{c} dx \\ dy \\ dz \end{array} \end{array}$$

$$= \int_D u \begin{vmatrix} 0 & 1 \\ v & u \end{vmatrix} + v(uv) \begin{vmatrix} v & u \\ 1 & 0 \end{vmatrix} + uv \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= \int_D -uv - u^2v^2 + uv = \int_0^1\int_0^1 -uv = -\frac{1}{4}$$

$S_u \times S_v = $ etc. $= (-v, -u, 1)$



(c) $S(u,v) = (u, v, uv)$, $F = (x, 0, z)$

$D = [0,1]^2$

$$JS = \begin{array}{c} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ v & u \end{bmatrix} \end{array} \begin{array}{l} dx \\ dy \\ dz \end{array}$$

$$S_u \times S_v = (-v, -u, 1)$$

flux: $\int_S \omega^F = \int_S x \, dy \wedge dz + 0 \, dz \wedge dx + z \, dx \wedge dy$

$$= \int_D -uv + 0 + uv = \int_D 0 = 0$$

Note: $F(S(u,v)) \cdot ((S_u \times S_v)(u,v)) = F(u, v, uv) \cdot (-v, -u, 1)$

$$= (u, 0, uv) \cdot (-v, -u, 1) = -uv + uv = 0$$

So $F(S(u,v))$ is perpendicular to the normal vector to S, i.e. F is flowing exactly along F. So the flux is 0.