Math 212 Announce Thursday Talk 

Quiz Let $\varphi : [0,1]^2 \longrightarrow \mathbb{R}^3$

$$(u,v) \longmapsto (u, v, 2u+v^2)$$

and let

$$\omega = y\,dx + x\,dy + x^2\,dz.$$

Verify Stokes' theorem, $\int_\varphi d\omega = \int_{\partial\varphi} \omega$, by directly computing both sides of the equation.

1. Compute $d\omega$.
2. Compute $\int_\varphi d\omega$
3. Compute $\varphi \circ \Delta^2_{i,\alpha}(t)$ $\forall i, \alpha$.
4. Compute $\int_{\partial\varphi} \omega$.

Solution/ $d\omega = dy \wedge dx + dx \wedge dy + 2x\,dx \wedge dz$

$$= -dx \wedge dy + dx \wedge dy + 2x\,dx \wedge dz$$

$$= 2x\,dx \wedge dz.$$

So $\int_\varphi d\omega = \int_{[0,1]^2} \varphi^* d\omega = \int_{[0,1]^2} \varphi^*(2x\,dx \wedge dz)$

$$= \int_{[0,1]^2} 2u\, du \wedge d(2u+v^2) = \int_{[0,1]^2} 2u\, du \wedge (2du + 2v dv)$$

$$= \int_{[0,1]^2} 4uv\, du \wedge dv = \int_0^1 \int_0^1 4uv\, du dv = \int_0^1 2v\, dv = 1.$$

On the other hand,

$(u, v, 2u+v^2)$

$$\partial \varphi = -\varphi \circ \Delta^2_{1,0} + \varphi \circ \Delta^2_{1,1} + \varphi \circ \Delta^2_{2,0} - \varphi \circ \Delta^2_{2,1},$$

here $\varphi \circ \Delta^2_{1,0}(t) = \varphi(0,t) = (0, t, t^2) =: C_1(t)$

$\varphi \circ \Delta^2_{1,1}(t) = \varphi(1,t) = (1, t, 2+t^2) =: C_2(t)$

$\varphi \circ \Delta^2_{2,0}(t) = \varphi(t,0) = (t, 0, 2t) =: C_3(t)$

$\varphi \circ \Delta^2_{2,1}(t) = \varphi(t,1) = (t, 1, 2t+1) =: C_4(t)$

Therefore,

$$\int_{\partial\varphi} w = -\int_{C_1} w + \int_{C_2} w + \int_{C_3} w - \int_{C_4} w$$

$$= -\int_{[0,1]} t\,d(0) + 0\,dt + 0^2 d(t^2)$$

$$+ \int_{[0,1]} t\,d(1) + 1\cdot dt + 1^2 d(t^2)$$

$$+ \int_{[0,1]} 0\,dt + t\,d(0) + t^2 d(2t)$$

$$- \int_{[0,1]} 1\,dt + t\,d(1) + t^2 d(2t+1)$$

$$= 0 + \int_0^1 (1+2t) + \int_0^1 2t^2 \quad - \int_0^1 (1+2t^2)$$

$$= \int_0^1 1+2t+2t^2-(1+2t^2) = \int_0^1 2t = t^2\Big|_0^1 = 1.$$

# Surface integrals

$S : D \to \mathbb{R}^3 \qquad D \subseteq \mathbb{R}^2$, $S$ continuously differentiable

$f : \mathbb{R}^3 \to \mathbb{R}$, continuous (weighting function)

**Def.** $$\int_S f = \int_S f \, dS = \int_D (f \circ S) \, |S_u \times S_v|$$

notation

length of the cross product, defined below

where $S = S(u,v)$, $S_u = \frac{\partial S}{\partial u}$, $S_v = \frac{\partial S}{\partial v}$.

If $f = 1$, we get the surface area

$$\text{area}(S) = \int_S 1 = \int_S dS = \int_D |S_u \times S_v| .$$

If $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$ are elements of $\mathbb{R}^3$,

then their cross product is

$$v \times w = \det \begin{vmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = (v_2 w_3 - w_2 v_3 \,,\, v_3 w_1 - v_1 w_3 \,,\, v_1 w_2 - v_2 w_1).$$

Relation with wedge product: $v \wedge w = (v_1 e_1 + v_2 e_2 + v_3 e_3) \wedge (w_1 e_1 + w_2 e_2 + w_3 e_3)$

$$= (v_2 w_3 - v_3 w_2)\, e_2 \wedge e_3 + (v_3 w_1 - v_1 w_3)\, e_3 \wedge e_1 + (v_1 w_2 - v_2 w_1)\, e_1 \wedge e_2$$

Remark Since $S_u$ and $S_v$ span the tangent space, we have the $S_u \times S_v$ is normal vector to the surface $S$, i.e. it's perpendicular to the surface.