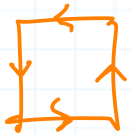
Math 212 Stokes' Thm.

Recall

Boundary

$I = [0,1]$. $\Delta^k : I^k \to I^k \subseteq \mathbb{R}^k$, $\Delta^k(u_1, \dots, u_k) = (u_1, \dots, u_k)$

$\Delta^k_{i,\alpha} : I^{k-1} \to I^k$, $\Delta^k_{i,\alpha}(u_1, \dots, u_{k-1}) = (u_1, \dots, u_{i-1}, \alpha, u_i, \dots, u_k)$



$$\partial \Delta^k = \sum_{i=1}^{k} \sum_{\alpha=0}^{1} (-1)^{i+\alpha} \Delta^k_{i,\alpha}$$

If $c : I^k \to \mathbb{R}^n$, then $\partial c = c \circ \partial \Delta^k = \sum_{i=1}^{k} \sum_{\alpha=0}^{1} (-1)^{i+\alpha} c \circ \Delta^k_{i,\alpha}$.

Properties of d and of pullbacks

$$d\varphi^* = \varphi^* d\omega \qquad \text{chain rule}$$

$$(\varphi\circ\psi)^* = \psi^*\varphi^* \qquad \text{functoriality}$$

$$\int_{\varphi\circ\psi} \omega = \int_{\psi} \varphi^*\omega \qquad \text{change of variables}$$

Def. of pullback

If $\omega = \sum_I f_I\, dx_I$,

then

$$\varphi^*\omega = \sum_I f_I\circ\varphi\, d\varphi_I$$

where $d\varphi_I = d\varphi_{i_1}\wedge\cdots\wedge d\varphi_{i_k}$.

Integration

$$\varphi: I^k \to \mathbb{R}^n,\quad \omega\in\Omega^k\mathbb{R}^n \implies \int_\varphi \omega = \int_{I^k}\varphi^*\omega$$

$$C = \sum_{s=1}^{t} v_s\,\varphi_{(s)},\quad \omega\in\Omega^k\mathbb{R}^n \implies \int_C \omega = \sum_{s=1}^{t} v_s\int_{\varphi_s}\omega.$$

Practice

$$\Delta^4_{2,0}(u_1,u_2,u_3) = (u_1, 0, u_2, u_3)$$

$$(\Delta^4_{2,0})^* dx_2\wedge dx_3\wedge dx_4 = d(0)\wedge du_2\wedge du_3 = 0$$

$$(\Delta^4_{2,1})^* dx_1\wedge dx_2\wedge dx_4 = du_1\wedge d(1)\wedge du_3 = 0$$

$$(\Delta^4_{2,1})^* dx_1\wedge dx_3\wedge dx_4 = du_1\wedge du_2\wedge du_3$$

Thm. If $C = \sum_{s=1}^{t} \nu_s Q_{(s)}$ is a k-chain in $\mathbb{R}^n$ and $\omega \in \Omega^{k-1}\mathbb{R}^n$, then

$$\int_C d\omega = \int_{\partial C} \omega.$$

omit

Pf/ Case 1 : $\omega = f\, dx_1 \wedge \cdots \wedge \overline{dx_j} \wedge \cdots \wedge dx_k$, $C = \Delta^k$.

$$\int_{\partial C} \omega = \sum_{i=1}^{k} \sum_{\alpha=0}^{1} (-1)^{i+\alpha} \int_{\Delta^k_{i,\alpha}} f\, dx_1 \wedge \cdots \wedge \overline{dx_j} \wedge \cdots \wedge dx_k$$

$(\Delta^k_{i,\alpha})^*(dx_1 \wedge \cdots \wedge \overline{dx_j} \wedge \cdots \wedge dx_k)$ $=0$ if $i \neq j$ since $d\alpha = 0$ for $\alpha = 0$ or 1.

$$= \sum_{a=0}^{1} (-1)^{j+\alpha} \int_{\Delta^k_{j,\alpha}} f\, dx_1 \wedge \cdots \wedge \overline{dx_j} \wedge \cdots \wedge dx_k$$

$$= (-1)^{j+1} \left[\int_{[0,1]^{k-1}} f(u_1, \ldots, u_{j-1}, 1, u_j, \ldots, u_{k-1}) - f(u_1, \ldots, u_{j-1}, 0, u_{j+1}, \ldots, u_k)\, du_1 \wedge \cdots \wedge du_{k-1} \right]$$

$$= (-1)^{j+1} \int_{[0,1]^{k-1}} \big(f(u_1, \ldots, u_{j-1}, 1, u_j, \ldots, u_{k-1}) - f(u_1, \ldots, u_{j-1}, 0, u_{j+1}, \ldots, u_k) \big)$$

By 1-variable version of FTC.

$$= (-1)^{j+1} \int_{[0,1]^{k-1}} \int_{t=0}^{1} \frac{\partial f}{\partial x_j} (u_1, \ldots, u_{j-1}, t, u_j, \ldots, u_{k-1})$$

$$\overset{\text{Fubini}}{=} (-1)^{j+1} \int_{[0,1]^k} \frac{\partial f}{\partial x_j} .$$

Whereas,

$$\int_C d\omega = \int_C \left(\sum_{i=1}^{k} \frac{\partial f}{\partial x_i} dx_i \right) \wedge dx_1 \wedge \cdots \wedge \overline{dx_j} \wedge \cdots \wedge dx_k$$

$$= \int_C \frac{\partial f}{\partial x_j} dx_j \wedge dx_1 \wedge \cdots \wedge \overline{dx_j} \wedge \cdots \wedge dx_k$$

$$= (-1)^{j+1} \int_C \frac{\partial f}{\partial x_j} dx_1 \wedge \cdots \wedge dx_j \wedge \cdots \wedge dx_k$$

$$= (-1)^{j+1} \int_{[0,1]^k} \frac{\partial f}{\partial x_j} .$$

So $\int_C d\omega = \int_{\partial C} \omega$ in this case.

Case 2: ω a general $k-1$ form, $C = \Delta^k$

In this case, $\omega = \sum_{j=1}^{k} \omega_j$ where $\omega_j = f_j \, dx_1 \wedge \cdots \wedge \overline{dx_j} \wedge \cdots \wedge dx_k$,

so

$$\int_{\partial C} \omega = \int_{\partial C} \sum_{j=1}^{k} \omega_j$$

$$= \sum_{j=1}^{k} \int_{\partial C} \omega_j$$

Case 1

$$= \sum_{j=1}^{k} \int_{C} d\omega_j$$

$$= \int_C d\left(\sum_{j=1}^{k} \omega_j\right) \qquad \text{(linearity of } \textstyle\int \text{ and } d\text{)}$$

$$= \int_C d\omega .$$

Case 3: $\omega \in \Omega^{k-1}\mathbb{R}^n$, $C = \varphi = \varphi \circ \Delta^k$ (singular k-cube in $\mathbb{R}^n$)

← general k-1 form

$$\int_C d\omega = \int_{[0,1]^k} C^* d\omega$$

$$C : I^k \xrightarrow{\Delta^k} I^k \xrightarrow{\varphi} \mathbb{R}^n$$

$$= \int_{[0,1]^k} (\varphi \circ \Delta^k)^* d\omega$$

$$= \int_{[0,1]^k} (\Delta^k)^* \circ \varphi^* \, d\omega$$

$$= \int_{\Delta^k} \varphi^* \, d\omega$$

$$= \int_{\Delta^k} d(\varphi^* \omega) \qquad \text{(chain rule)}$$

Case 2

$$= \int_{\partial \Delta^k} \varphi^* \omega$$

$$= \sum_{i=1}^{k} \sum_{\alpha=0}^{1} (-1)^{i+\alpha} \int_{\Delta^k_{i,\alpha}} \varphi^* \omega$$

$$= \sum_{i=1}^{k} \sum_{\alpha=0}^{1} (-1)^{i+\alpha} \int_{\varphi \circ \Delta^k_{i,\alpha}} \omega$$

$$= \int_{\sum_{i=1}^{k} \sum_{\alpha=0}^{1} (-1)^{i+\alpha} \varphi \circ \Delta^k_{i,\alpha}} \omega$$

$$= \int_{\partial \varphi} \omega .$$

Case 4: $\omega \in \Omega^{k-1}\mathbb{R}^n$, $C = \sum_{s=1}^{t} \nu_s \phi_{(s)}$ (general chain)

$$\int_C d\omega = \sum_{i=1}^{t} \nu_s \int_{\phi_{(s)}} d\omega$$

Case 3

$$= \sum_{i=1}^{t} \nu_s \int_{\partial\phi_{(s)}} \omega$$

$$= \int_{\sum_{i=1}^{t} \nu_s \partial\phi_{(s)}} \omega$$

$$= \int_{\partial C} \omega .$$

$\square$