Math 212

Last time $\varphi : \mathbb{R}^n \to \mathbb{R} \Rightarrow d\varphi = \sum \frac{\partial \varphi}{\partial x_i} dx_i = \omega_{\nabla\varphi}$

φ is a potential for the gradient vector field $\nabla \varphi = \left(\frac{\partial \varphi}{\partial x_1}, \dots, \frac{\partial \varphi}{\partial x_n} \right)$.

(Note: Given a potential φ, we have that φ + constant is also a potential for the same vector field.)

Stokes': If $c : [a,b] \to \mathbb{R}^n$, then

$$\int_c \nabla\varphi \cdot dc = \int_c \omega_{\nabla\varphi} = \int_c d\varphi \overset{\text{Stokes'}}{=} \int_{\partial c} \varphi = \varphi(c(b)) - \varphi(c(a))$$

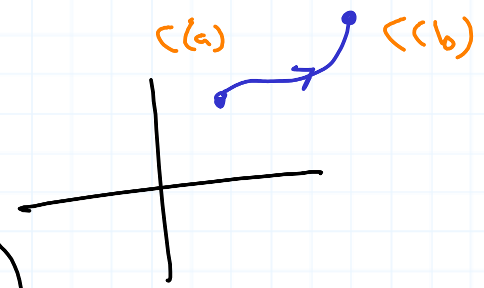
The flow is given by the change in potential in the case of a gradient vector field.

Example $F = -\alpha \frac{\vec{r}}{r^2}$ where $\alpha \in \mathbb{R}_{>0}$, $\vec{r} = \frac{(x_1, \dots, x_n)}{\sqrt{x_1^2 + \dots + x_n^2}}$, $r = |(x_1, \dots, x_n)| = \sqrt{x_1^2 + \dots + x_n^2}$

A potential for F is $\Phi = \frac{\alpha}{r} = \alpha (x_1^2 + \dots + x_n^2)^{-\frac{1}{2}}$.

If $n = 3$ and $\alpha = GmM$, we get Newton's gravitational force between masses m, M where m sits at the origin and M sits at $r = (x_1, \dots, x_n)$. The gravitational potential is defined to be $-\Phi$. Note the sign change. Thus, as M moves away from m, its gravitational potential <u>increases</u> (becomes less negative) whereas the potential Φ decreases.

If M travels along a curve $C : [a, b] \to \mathbb{R}^n$ with $|C(b)| > |C(a)|$ then the work done on M by F is



$$\int_C F \cdot dC = \int_C \omega_F = \int_C \omega_{\nabla\Phi} = \Phi(C(b)) - \Phi(C(a)) < 0$$

(since M is moving generally in a direction opposite to F).

But $\Phi(c(b)) - \Phi(c(a)) = -\left([-\Phi(c(b))] + [-\Phi(c(a))]\right) > 0$

$-\Phi$ = gravitational potential

So the gravitational potential has increased. So the gravitational potential tells you how much work you must do against the force.

Conservative Forces

Def. Let F be a vector field on $\mathbb{R}^n$. We say F has the **path independence property** if the flow of F along a curve in $\mathbb{R}^n$ only depends on the endpoints of the curve. We say F is a **conservative** vector field if F has a potential, i.e., if F is a gradient vector field.

Thm. The following are equivalent for a vector field F on $\mathbb{R}^n$:

$y = mx + b$

$0 = m + b$

$2 = m + 2b$

1) F has the path independence property.

2) F is conservative.

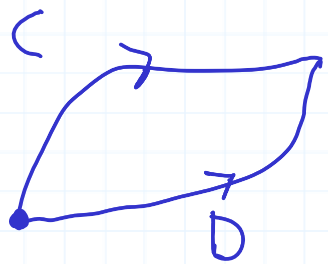
3) The flow of F along every closed curve is 0.

Pf/ 1) $\Rightarrow$ 2) (partial) Fix a point $p \in \mathbb{R}^n$. For $x \in \mathbb{R}^n$, pick any path C_x from p to x, and define $Q(x) = \int_{C_x} F \cdot dC_x$.

One may show $\nabla Q = F$.

2) $\Rightarrow$ 3) Stokes' (See above)

3) $\Rightarrow$ 1) Let C, D be curves in $\mathbb{R}^n$ with the same initial and endpoints. For ease of notation, assume both curves have domain $[0,1]$



Consider the curve "$C - D$", i.e., first C, then D traveled in reverse. This is a closed curve, so by hypothesis $0 = \int_{C-D} \omega_F = \int_C \omega_F - \int_D \omega_F$

$\Rightarrow \int_C \omega_F = \int_D \omega$, as required.

More formally, let

$$E(t) = \begin{cases} C(2t), & 0 \le t \le \frac{1}{2} \\ D(2-2t), & \frac{1}{2} \le t \le 1. \end{cases}$$

Then E is a closed curve: $E(0) = C(0) = D(0) = E(1)$.

So $0 = \int_E \omega_F = \int_0^1 (F \circ E) \cdot E' = \int_0^{\frac{1}{2}} \underbrace{[(F \circ C)(2t)] \cdot [2\, C'(2t)]\, dt}_{1}$

$+ \int_{\frac{1}{2}}^1 \underbrace{[(F \circ D)(2-2t)] \cdot [-2\, D'(2-2t)]\, dt}_{2}$

Substitute $s=2t$ in the first integral, ①, and $s=2-2t$ in the second integral, ②. Then ① becomes $\int_0^1 (F\circ C)(t)\cdot C'(t)\,dt$ $=\int_C \omega_F$ and ② becomes $\int_1^0 (F\circ D)(t)\cdot D'(t)\,dt$, i.e.,

$-\int_0^1 (F\circ D)(t)\cdot D'(t)\,dt = -\int_D \omega_F$. We've thus shown $0=\int_C \omega_F - \int_D \omega_F$. $\square$

Remark Suppose $F=\nabla\varphi$. We've seen the flow of F along a curve C is $\int_C \omega_F = \int_a^b (F\circ C)\cdot C' = \int_a^b (F\circ C)\cdot \frac{C'}{|C'|}\,|C'|$

$$= \int_a^b \underbrace{\left[\nabla\varphi(c(t))\cdot \frac{C'(t)}{|C'(t)|}\right]}_{\text{Directional derivative of } \varphi \text{ in the direction of } \frac{C'(t)}{|C'(t)|}\text{, i.e. in the direction of } C.} |C'| \overset{\text{Stokes}}{=} \varphi(c(b)) - \varphi(c(a)) = \text{change in potential.}$$

Thus, the change in φ along C is the integral of the rate of change of φ in the direction of C.