 ☆ Announce talk

Quiz

1. Let Δ^k be the standard k-cube. What is the definition of $\Delta^k_{i,\alpha}$?

2. Let $I = [0,1]$, and let $\varphi : I^k \to \mathbb{R}^n$ be a k-surface in $\mathbb{R}^n$. What is the definition of $\partial\varphi$?

Last time $C : [a,b] \to \mathbb{R}^n$, $F : \mathbb{R}^n \to \mathbb{R}^n$

curve in $\mathbb{R}^n$ vector field in $\mathbb{R}^n$

$$\omega_F = \sum_{i=1}^n F_i \, dx_i \quad , \quad \int_C F \cdot dC := \int_C F \cdot \vec{t} = \int_C \omega_F$$

flow form flow of F along C

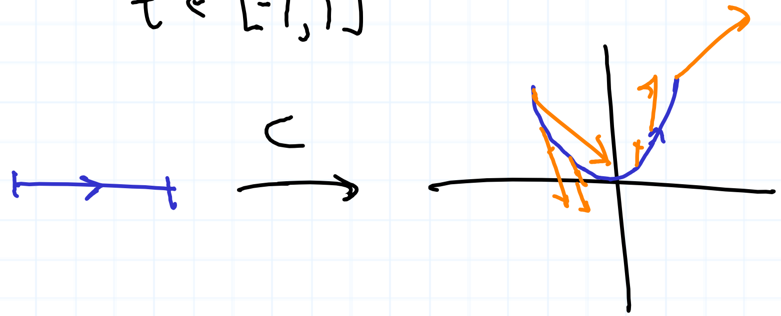
Useful formula $\int_C \omega_F = \int_a^b (F \circ C) \cdot C'$ [We showed this last time.]

dot product

Volunteers

Example $F(x,y) = (y^2, x)$, $C(t) = (t, t^2)$ $\quad t \in [-1,1]$

(Use Sage to draw relevant picture.)



Two methods

(a) Directly: $\int_C \omega_F = \int_C y^2\,dx + x\,dy = \int_{[-1,1]} t^4\,d(t) + t\,d(t^2)$

$$= \int_{[-1,1]} t^4\,dt + 2t^2\,dt = \int_{[-1,1]} (t^4+2t^2)\,dt = \int_{-1}^{1} t^4+2t^2 = \frac{t^5}{5} + \frac{2}{3}t^3 \Big|_{-1}^{1}$$

$$= \frac{1}{5} + \frac{2}{3} - \left[-\frac{1}{5} - \frac{2}{3}\right] = \frac{26}{15}.$$

(b) Using formula $\int_C \omega_F = \int_{-1}^{1} (F \circ C) \cdot C' = \int_{-1}^{1} F(t,t^2) \cdot (1,2t)$

$$= \int_{-1}^{1} (t^4, t) \cdot (1, 2t) = \int_{-1}^{1} t^4 + 2t^2 = \frac{26}{15}.$$

Note that this method is faster.

Stokes'

Let $\varphi : \mathbb{R}^n \to \mathbb{R}$. Then $d\varphi = \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i} dx_i$, which is the flow form for the gradient $\nabla \varphi = \left(\frac{\partial \varphi}{\partial x_1}, \ldots, \frac{\partial \varphi}{\partial x_n}\right)$. So $^{*}d\varphi = \omega_{\nabla \varphi}^{*}$.

Recall

Main facts about the gradient: ① $\nabla \varphi$ is perpendicular to level sets $\varphi = \text{constant}$.

② $\nabla \varphi$ points in the direction of quickest growth of φ.

③ The length of $\nabla \varphi$ is the rate of change of φ in the direction of quickest growth.

④ If $v \in \mathbb{R}^n$ is a unit vector, then $\nabla \varphi \cdot v$ is the directional derivative — the rate of change of φ in the direction of φ.

Given a curve $C : [a,b] \to \mathbb{R}^n$, the flow of $\nabla \varphi$ along C is

$$\int_C \omega_{\nabla \varphi} = \int_C d\varphi \overset{\text{Stokes'}}{=} \int_{\partial C} \varphi = \varphi(C(b)) - \varphi(C(a)).$$

(Recall $\partial C = -C \circ \Delta^1_{1,a} - C \circ \Delta^1_{1,b}$ where $\Delta^1_{1,a} : () \to a$ and $\Delta^1_{1,b} : () \to b$ so $C \circ \Delta^1_{1,a} : () \to C(a)$ and $C \circ \Delta^1_{1,b} : () \to C(b)$. Also, integrating over a 0-chain is given by evaluating.)

Def. A vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ is a gradient vector field if $F = \nabla \varphi$ for some $\varphi : \mathbb{R}^n \to \mathbb{R}$. In this case φ is called a potential for φ.

Just above, we used Stokes' theorem to prove:

Thm. If F is a gradient vector field with potential φ, and $C : [a,b] \to \mathbb{R}^n$, then the flow of F along C is given by the change in potential between the endpoints of C:

$$\int F \cdot dC = \varphi(C(b)) - \varphi(C(a)).$$

Example $F = (1, 2y)$ has potential $\varphi = x + y^2$.

(a) $C(t) = (t, t^n)$ for any $n \neq 0$, $0 \leq t \leq 1$.

Then $\int_C F \cdot dC = \int_0^1 (F \circ C) \cdot C' = \int_0^1 F(t, t^n) \cdot (1, nt^{n-1})$

$= \int_0^1 (1, 2t^n) \cdot (1, nt^{n-1}) = \int_0^1 1 + 2nt^{2n-1} = t + t^{2n} \Big|_0^1 = 2$

$$= \varphi(c(1)) - \varphi(c(0)) = \varphi(1,1) - \varphi(0,0) = 2 - 0 = 2.$$

In fact $\int_C F \cdot dc = 2$ for any curve starting at 0 and ending at 1.

Def. If the vector field F is thought of as a force field, the the **work** done by F on a particle traveling along the path parametrized by C is the flow of F along C:

$$\text{work} = \int F \cdot dc = \int_C \omega_F$$

where ω_F is the flow form.

If the force F has a potential, the work done by F is given by the change in potential.