Math 212

☆ return Midterms

☆ tomorrow's quiz

Last time

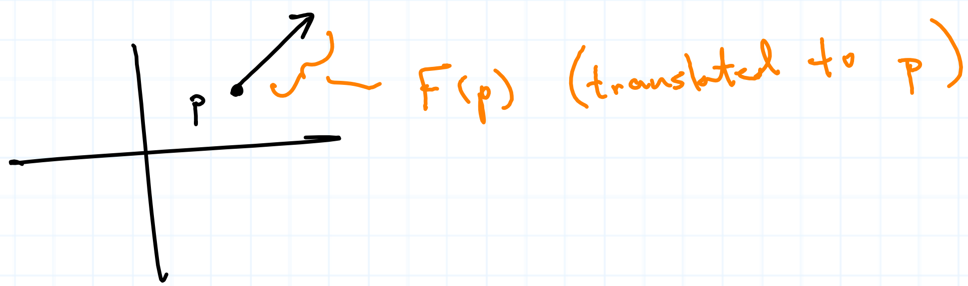
$C : [a,b] \to \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$

path integral : $\int_C f := \int_a^b f \circ c \, |c'|$ = weighted length of C.

Today: flow of a vector field along a curve

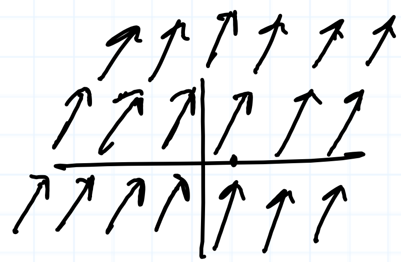
Def. A vector field on $\mathbb{R}^n$ is a function $F : \mathbb{R}^n \to \mathbb{R}^n$.

picture:

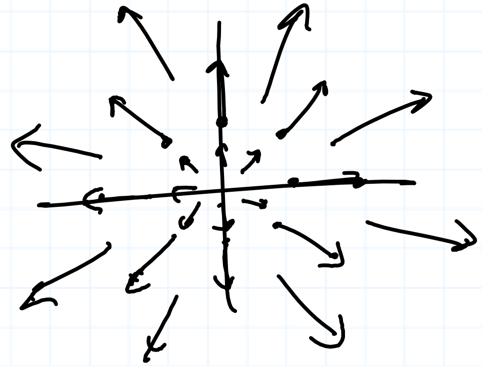


Examples

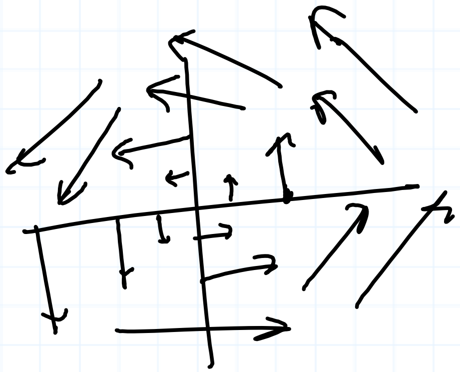
(1) $F(x,y) = (1,2)$



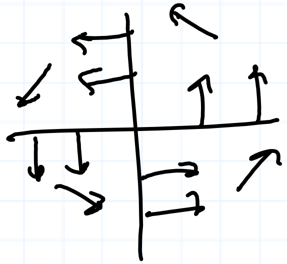
(2) $F(x,y) = (x,y)$



(3) $F(x,y) = (-y,x)$



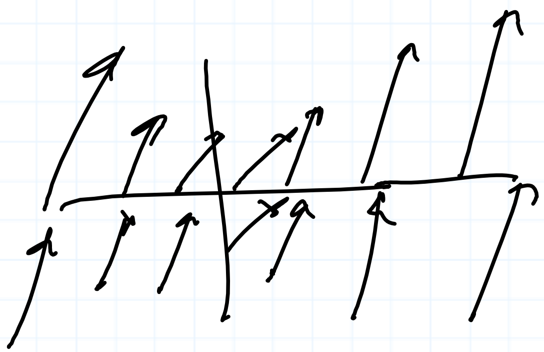
(4) $F(x,y) = \underbrace{\overset{\text{scalar}}{\overset{\downarrow}{\frac{1}{\sqrt{x^2+y^2}}}}(-y,x)}_{\text{unit length}}$



$F: \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2$

(not defined at $(0,0)$)

(5) $F(x,y) = (1, x^2)$



(See Sage)

③

Def. $C:[a,b]\to\mathbb{R}^n$ (curve), $F=(F_1,F_2,\dots,F_n):\mathbb{R}^n\to\mathbb{R}^n$ (vector field)

The flow of F along C is $\int_C \omega_F$ where

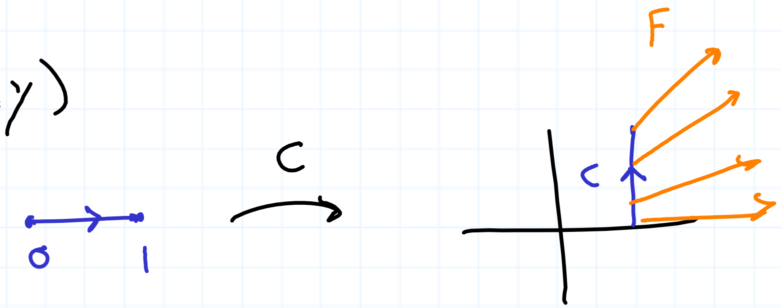
ω_F is the flow form : $\omega_F = F_1\,dx_1+\cdots+F_n\,dx_n$.

other notation: $\int_C F\cdot\vec{t}$ or $\int_C F\cdot dC$

or $\oint_C F\cdot dC$ if C is closed, i.e. $C(a)=C(b)$.

Example $F(x,y)=(x,y)$

(1) $C(t)=(1,t)$, $0\le t\le 1$ ($x = 1$, $y = t$)

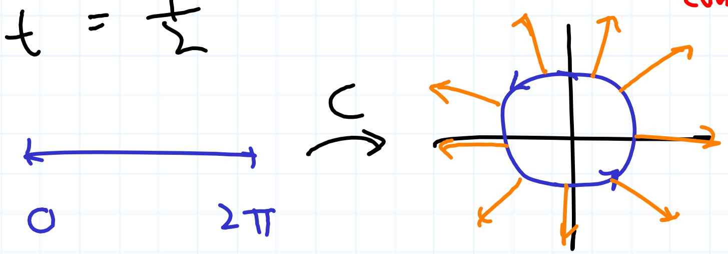


mixing up notation

$$\int_C F\cdot d\vec{t} = \int_C \omega_F = \int_C x\,dx+y\,dy = \int_{[0,1]} \underset{\text{pullback}}{C^*}(x\,dx+y\,dy)$$

$$= \int_{[0,1]} 1\cdot d(1) + t\,d(t) = \int_{[0,1]} t\,dt = \int_0^1 t = \frac{1}{2}$$

(2) $C(t)=(\cos t,\sin t)$, $0\le t\le 2\pi$ ($x = \cos t$, $y = \sin t$)



✱ No component of F along C

$$\int_C F\cdot dC = \int_C \omega_F = \int_C x\,dx+y\,dy = \int_{[0,2\pi]} \cos t\, d(\cos t) + \sin t\, d(\sin t)$$

$$= \int_{[0,2\pi]} -\cos t\sin t\,dt + \sin t\cos t\,dt = \int_{[0,2\pi]} 0\,dt = \int_0^{2\pi} 0 = 0.$$

Explanation of formula

$$C : [a,b] \to \mathbb{R}^n \qquad C = (\overset{x_1}{\overset{\shortparallel}{C_1}}, \dots, \overset{x_n}{\overset{\shortparallel}{C_n}})$$

$$F : \mathbb{R}^n \to \mathbb{R}^n \qquad F = (F_1, \dots, F_n)$$

$$\int_C \omega_F = \int_C F_1 dx_1 + \cdots + F_n dx_n$$

$$= \int_{[a,b]} C^*(F_1 dx_1 + \cdots + F_n dx_n)$$

$$= \int_{[a,b]} F_1 \circ C \; dC_1 + \cdots + F_n \circ C \; dC_n$$

$$= \int_{[a,b]} (F_1 \circ C) C_1' dt + \cdots + (F_n \circ C) C_n' dt$$

$$= \int_a^b (F_1 \circ C) C_1' + \cdots + (F_n \circ C) C_n'$$

dot product

$$= \int_a^b (F_1 \circ C, \dots, F_n \circ C) \cdot (C_1', \dots, C_n')$$

$$= \int_a^b \underset{\nearrow}{(F\circ C)\cdot}\ C'$$

dot products

$$[a,b] \xrightarrow{C} \mathbb{R}^n \xrightarrow{F} \mathbb{R}^n$$

$$(F\circ C)(t) = F(C(t)) = (F_1(C(t)),\dots,F_n(C(t)))$$

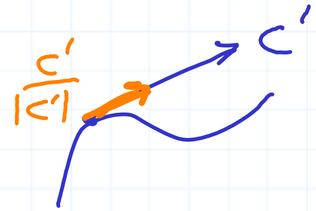
$$= ((F_1\circ C)(t),\dots,(F_n\circ C)(t))$$

$$= \int_a^b \underbrace{(F\circ C)\cdot \frac{C'}{|C'|}}_{f}\ |C'|$$

$$= \int_a^b f\,|C'|$$

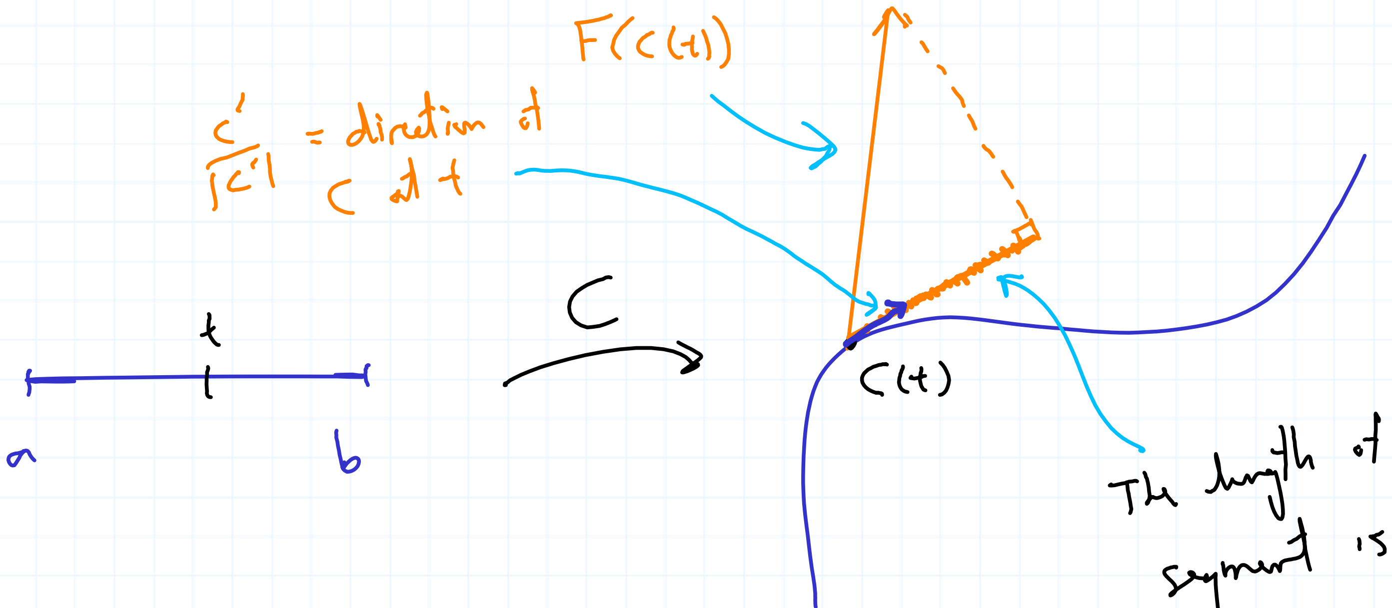
$= \text{length of } C \text{ weighted by } f = (F\circ C)\cdot \underbrace{\frac{C'}{|C'|}}_{\text{unit vector in tangent direction}}$

Note: $\left[(F\circ C)\cdot \frac{C'}{|C'|}\right](t)$ is the component of F at $C(t)$ in the unit tangent direction, $\dfrac{C'(t)}{|C'(t)|}$.



☆ Thus, $\int_C \omega_F$ is the length of C where each point of C

is weighted by the component of F in the direction of C at that point.



The length of this line segment is $(F \circ C) \cdot \frac{C'}{|C'|}$ at t.