Math 212

Div, grad, curl, flow, flux – Applications of differential forms

Let $K \subseteq \mathbb{R}^n$ and $f: K \to \mathbb{R}$. At the beginning of the semester, we defined $\int_K f$ and interpreted it as weighted n-dimensional volume.

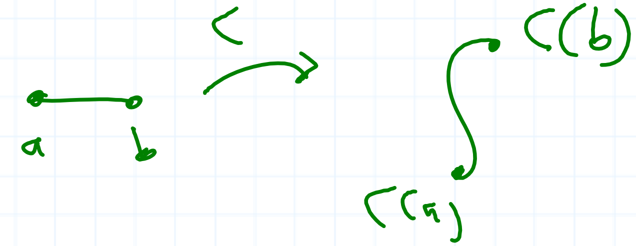
If K is the unit interval in $\mathbb{R}$, its volume (1-dimensional) would be 1, but if $K = \{(x,0) : x \in [0,1]\}$, i.e., if K is the same interval embedded in $\mathbb{R}^2$, its volume (2-dimensional) is now 0. We need the notion of the k-dimensional area of a k-dimensional object embedded in $\mathbb{R}^n$.

Path integrals (= curve integrals = line integrals)

Motivation for definition Let $C : [a,b] \to \mathbb{R}^n$ be a C^1 curve

the derivatives of its component functions are continuous

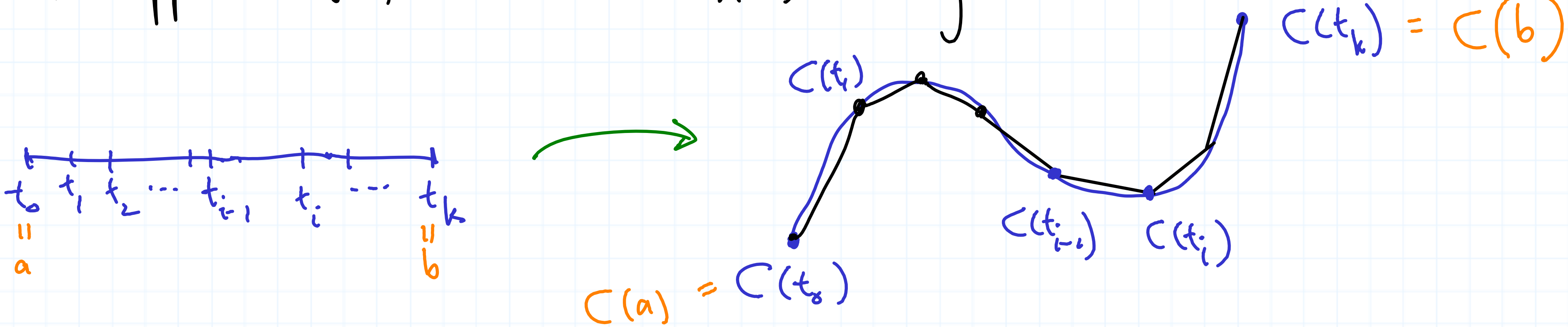
Think of C as parametrizing a piece of wire.



Let f be an $\mathbb{R}$-valued function defined on the image of C. Think of it as providing a density to each point of the curve.

Problem: Define the total weight of the curve.

Main idea: Approximate the curve with line segments:



Approximate the density of the i^{th} line segment as $f(C(t_i))$.

Then,

$$\text{weighted length of } C \approx \sum_{i=1}^{k} f(C(t_i)) \overbrace{\left| C(t_i) - C(t_{i-1}) \right|}^{\text{length of } i^{th} \text{ line segment}}$$

weight / length.

$$= \sum_{i=1}^{k} f(C(t_i)) \left| \frac{C(t_i) - C(t_{i-1})}{t_i - t_{i-1}} \right| (t_i - t_{i-1})$$

$$\approx \sum_{i=1}^{k} \underbrace{f(C(t_i)) \left| C'(t_i) \right|}_{g(t_i)} (t_i - t_{i-1})$$

$$\approx \sum_{i=1}^{k} g(t_i)(t_i - t_{i-1}) \approx \int_a^b g$$

$$\approx \int_a^b f(C(t)) \left| C'(t) \right|$$

$$= \int_a^b f \circ C \, |C'| .$$

Reminder

$$C'(t) := \lim_{s \to t} \frac{C(t) - C(s)}{t - s}$$

$$= (C_1'(t), \dots, C_n'(t))$$

$=$ velocity of C at time t.

Def. The path integral of f along C is

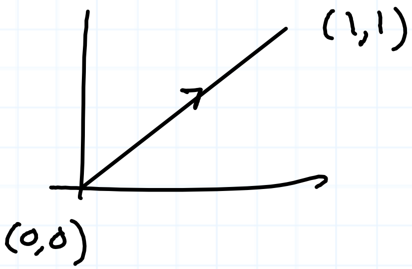
$$\int_C f := \int_a^b f\circ C\,|C'|.$$

Alternate notation: $\int_C f\,dC$.

Choose 3 "volunteers" to do these at the board.

Examples

(1)



(a) $C(t) = (t,t) \quad 0 \le t \le 1$

(b) $D(t) = (t^2, t^2) \quad 0 \le t \le 1$

(c) $E(t) = (\sin t, \sin t) \quad 0 \le t \le \frac{\pi}{2}$

$f(x,y) = 1 \quad \forall\, (x,y).$

(a) $\int_C f = \int_0^1 \underbrace{f\circ C}_{1}\,|C'| = \int_0^1 |(1,1)| = \int_0^1 \sqrt{2} = \sqrt{2}$

Note: $|t| = t$ on $[0,1]$

(b) $\int_D f = \int_0^1 f\circ D\,|D'| = \int_0^1 |(2t,2t)| = \int_0^1 \sqrt{4t^2+4t^2} = \int_0^1 \sqrt{8}\,t = \frac{\sqrt{8}}{2} t^2 \Big|_0^1 = \sqrt{2}.$

(c) $\int_E f = \int_0^{\frac{\pi}{2}} |E'| = \int_0^{\frac{\pi}{2}} |(\cos t, \cos t)| = \int_0^{\frac{\pi}{2}} \sqrt{\cos^2 t + \cos^2 t} = \int_0^{\frac{\pi}{2}} \sqrt{2}\,\cos t = \sqrt{2}\,\sin t \Big|_0^{\frac{\pi}{2}} = \sqrt{2}.$

$\leftarrow |\cos t| = \cos t$ on $[0,1]$

What about $\tilde{E}(t) = (\sin t, \sin t) \quad 0 \le t \le \pi$?

$$\int_{\tilde{E}} f = \int_0^{\pi} |E'| = \int_0^{\pi} |(\cos t, \cos t)| = \int_0^{\pi} \sqrt{2}\,|\cos t|.$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{2}\cos t + \int_{\frac{\pi}{2}}^{\pi} \sqrt{2}\,(-\cos t)$$

$|\cos t| = -\cos t$ on $[\frac{\pi}{2}, \pi]$

$$= \sqrt{2}\sin t\Big|_0^{\frac{\pi}{2}} - \sqrt{2}\Big|_{\pi/2}^{\pi} \sin t = 2\sqrt{2}.$$

(2) $C(t) = (\cos t, \sin t) \quad 0 \le t \le 2\pi, \quad f = 1$

$$\int_C f = \int_0^{2\pi} |(-\sin t, \cos t)| = \int_0^{2\pi} \sqrt{\cos^2 t + \sin^2 t} = \int_0^{2\pi} 1 = 2\pi.$$

Note: if $0 \le t \le 4\pi$, then $\int_C f = 4\pi = 2(2\pi)$.

(3) $C(t) = (t, t), \quad 0 \le t \le 1; \quad f(x,y) = x.$

Should $\int_C f$ be greater or less than $\int_C 1$?