Math 212

Boundary

Announce math talk

standard k-cube

$$\Delta^k : I^k \to I^k, \quad p \mapsto p \qquad , \qquad \Delta^k_{i,\alpha} : I^{k-1} \to I^k$$

$$(x_1, \dots, x_{k-1}) \mapsto (x_1, \dots, x_{i-1}, \alpha, x_i, \dots, x_{k-1})$$

$$\partial \Delta^k = \sum_{i=1}^{k} \sum_{\alpha=0}^{1} (-1)^{i+\alpha} \Delta^k_{i,\alpha}$$

set i^{th} variable of Δ^k equal to α

singular k-chain in $\mathbb{R}^n$

$$\phi : I^k \to \mathbb{R}^n \implies \partial \phi = \phi \circ \partial \Delta^k = \sum_{i=1}^{k} \sum_{\alpha=0}^{1} (-1)^{i+\alpha} \underbrace{\phi \circ \Delta^k_{i,\alpha}}$$

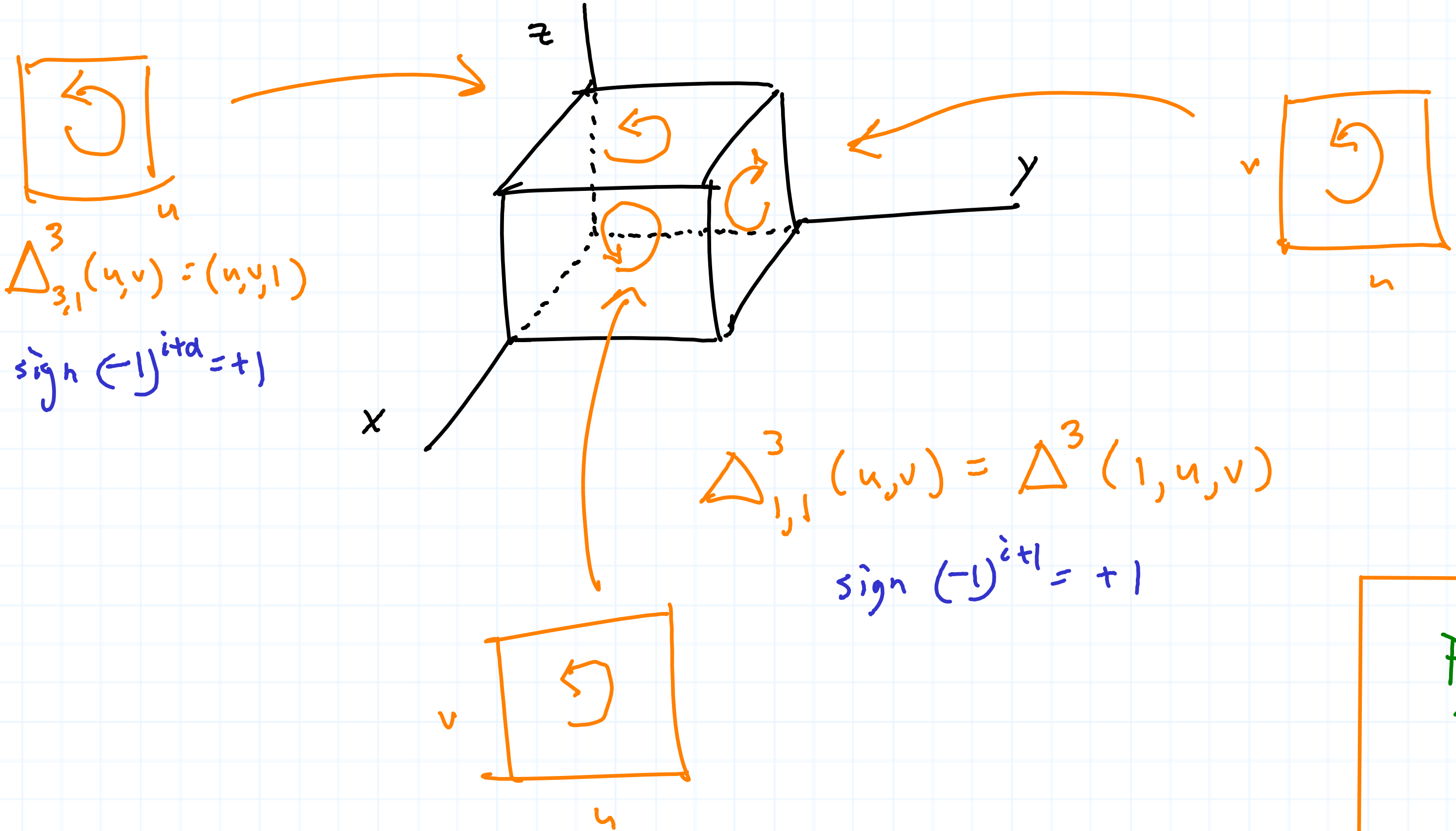
set i^{th} variable of ϕ equal to α.

Chains

$$c = \sum_{s=1}^{\ell} \nu_s \phi_{(s)} \implies \partial c = \sum_{s=1}^{\ell} \nu_s \, \partial \phi_{(s)}$$

Example: Boundary of the standard 3-cube

$$\partial \Delta^3 = \sum_{i=1}^{3} \sum_{\alpha=0}^{1} (-1)^{i+\alpha} \, \Delta^3_{i,\alpha}$$



$\Delta^3_{3,1}(u,v) = (u,v,1)$

sign $(-1)^{i+\alpha} = +1$

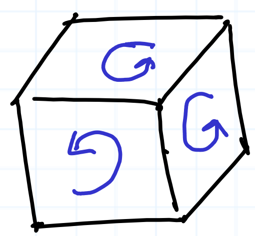
$\Delta^3_{2,1}(u,v) = (u,1,v)$

sign $(-1)^{i+\alpha} = -1$

$\Delta^3_{1,1}(u,v) = \Delta^3(1,u,v)$

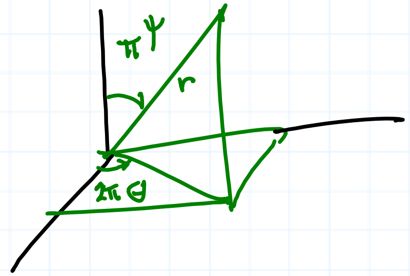
sign $(-1)^{i+1} = +1$

Picture of boundary

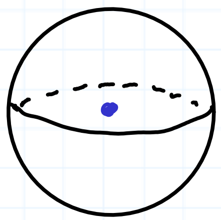


Note the "cancellation" of ↻ along the edges.

Example: Boundary of a sphere



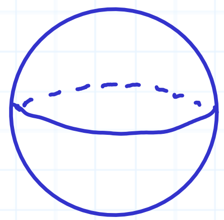
$$\varphi(r, \theta, \psi) = (r \cos 2\pi\theta \sin \pi\psi,\ r \sin 2\pi\theta \sin \pi\psi,\ r \cos \pi\psi)$$



$r = 0$

$i = 1, \alpha = 0$

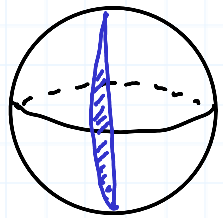
$-$



$r = 1$

$i = 1, \alpha = 1$

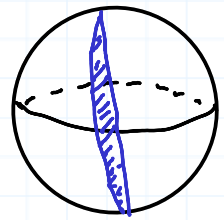
$+$



$\theta = 0$

$i = 2, \alpha = 0$

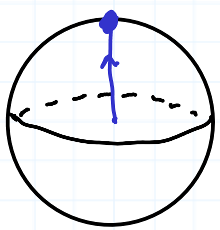
$+$



$\theta = 1$

$i = 2, \alpha = 1$

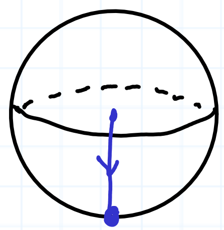
$-$



$\psi = 0$

$i = 3, \alpha = 0$

$-$

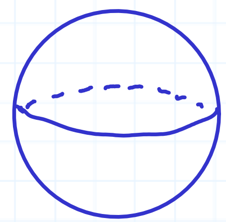


$\psi = 1$

$i = 3, \alpha = 1$

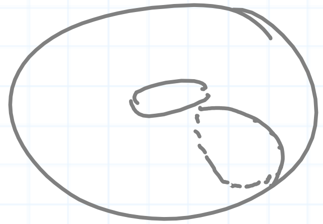
$+$

Integration only sees



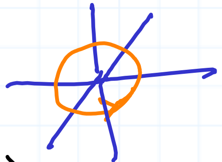
Example: Boundary of a donut

$$D(r,\theta,\psi) = \big((a+rb\cos 2\pi\theta)\cos 2\pi\psi,\ (a+rb\cos 2\pi\theta)\sin 2\pi\psi,\ rb\sin 2\pi\theta\big)$$



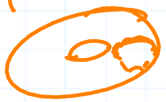
$$\partial D = ?$$

$\boxed{-}$ $r=0$



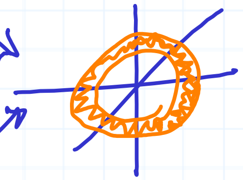
$D\circ\Delta^3_{1,0}(\theta,\psi) = (a\cos 2\pi\psi,\ a\sin 2\pi\psi,\ 0)$

$D\circ\Delta^3_{1,1}(\theta,\psi) = \text{etc.}$ $\boxed{+}$ $r=1$



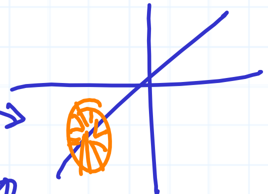
$D\circ\Delta^3_{2,0}(r,\psi) = \text{etc.}$ $\boxed{+}$ $\theta=0$

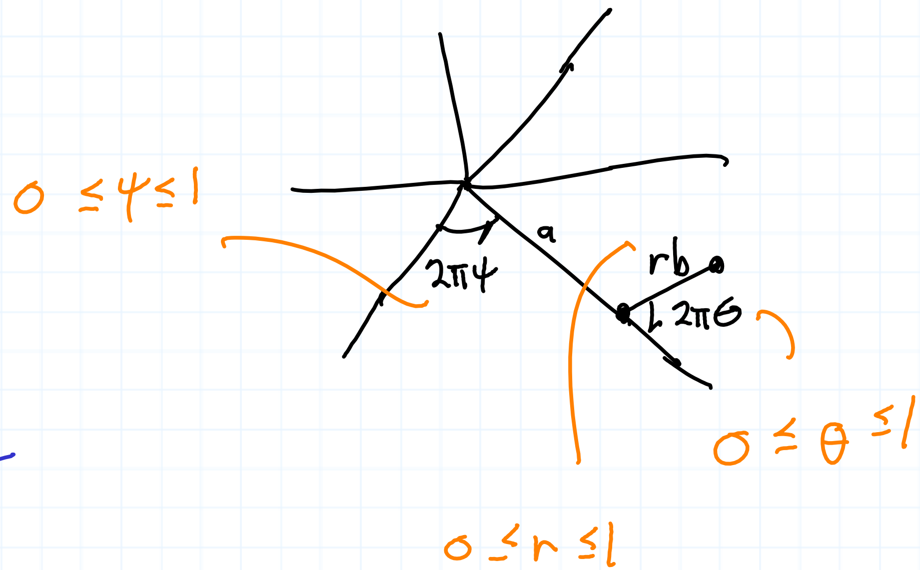
$D\circ\Delta^3_{2,1}(r,\psi) = \text{etc.}$ $\boxed{-}$ $\theta=1$



$D\circ\Delta^3_{3,0}(r,\theta) = \text{etc.}$ $\boxed{-}$ $\psi=0$

$D\circ\Delta^3_{3,1}(r,\theta) = \text{etc.}$ $\boxed{+}$ $\psi=1$





For a 2-form ω in $\mathbb{R}^3$,

$$\int_{\partial \mathcal{D}} \omega = \sum_{i=1}^{3} \sum_{\alpha=0}^{1} (-1)^{i+\alpha} \int_{\mathcal{D} \circ \Delta^3_{i,\alpha}} \omega$$

$$= \int_{\mathcal{D} \circ \Delta^3_{1,1}} \omega .$$

(All the other terms are zero or cancel.)