Math 212

* Midterm on Friday

* Warm-up problem $\quad \omega = x\,dx\wedge dy + dy\wedge dz \in \Omega^2 \mathbb{R}^3$

$$\varphi : [0,1]^2 \longrightarrow \mathbb{R}^3$$

$$(u,v) \longmapsto \left(\underset{\substack{\shortparallel \\ x}}{2u+v},\ \underset{\substack{\shortparallel \\ y}}{u+3v},\ \underset{\substack{\shortparallel \\ z}}{uv} \right)$$

Compute $\int_\varphi \omega$.

Solution/ $\displaystyle \int_\varphi \omega = \int_{[0,1]^2} \varphi^*\omega = \int_{[0,1]^2} (2u+v)\, d(2u+v)\wedge d(u+3v) +$

$\displaystyle + \; d(u+3v)\wedge d(uv) = \int_{[0,1]^2} (2u+v)\,(2du+dv)\wedge(du+3dv) +$

$\displaystyle (du+3dv)\wedge(v\,du+u\,dv) = \int_{[0,1]^2} (2u+v)\,[5\,du\wedge dv] + (u-3v)\,du\wedge dv$

$\displaystyle = \int_{[0,1]^2} 11u+2v \; du\wedge dv = \int_0^1\!\!\int_0^1 11u+2v\; du\,dv = \frac{13}{2}.$

A quicker way to compute the pullback:

$$J\varphi = \begin{array}{cc} \begin{bmatrix} 2 & 1 \\ 1 & 3 \\ v & u \end{bmatrix} & \begin{matrix} x \\ y \\ z \end{matrix} \\ \begin{matrix} u & v \end{matrix} & \end{array} \qquad \varphi^* dx \wedge dy = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 5$$

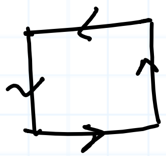
$$\varphi^* dy \wedge dz = \begin{vmatrix} 1 & 3 \\ v & u \end{vmatrix} = u - 3v.$$

Stokes' thm. example $\qquad \omega = xy\, dx$

1. Integrate $d\omega$ on $[0,1]^2$.

Solution / $d\omega = d(xy) \wedge dx = [y\,dx + x\,dy] \wedge dx = -x\, dx \wedge dy$;

so $\displaystyle\int_{[0,1]^2} d\omega = \int_0^1 \int_0^1 -x\, dx\, dy = -\frac{1}{2}$.

2. Integrate ω along the "boundary" of the square .

[Part of the problem here is to make sense of the question, which we will do in general later.]

Solution/ Parametrize each of the four sides of the square separately:

$\varphi_1(t) = (t, 0)$, $\varphi_2(t) = (1, t)$, $\varphi_3(t) = (1-t, 1)$, $\varphi_4(t) = (0, 1-t)$

x, y $\quad x, y \quad x, y \quad x, y$

for $0 \leq t \leq 1$. Then

$w = xy\,dx$

$$\int_{\partial \square} w = \sum_{i=1}^{4} \int_{\varphi_i} w = \sum_{i=1}^{4} \int_{[0,1]} \varphi_i^* w$$

$$= \int_{[0,1]} \underbrace{t \cdot 0\, dt}_{0} + \int_{[0,1]} \underbrace{1 \cdot t\, d(1)}_{0} + \int_{[0,1]} (1-t) \cdot 1\, d(1-t) + \int_{[0,1]} \underbrace{0\,(1-t)\, d0}_{0}$$

$$= \int_{[0,1]} (1-t)\left[d(1) + d(-t)\right] = \int_{[0,1]} (1-t)(-dt) = \int_0^1 t - 1 = -\frac{1}{2}.$$

Punchline $\int_\phi dw = \int_{\partial\phi} w$.

The boundary in the example above could be written as $\Phi_1 + \Phi_2 + \Phi_3 + \Phi_4$ where the addition is only formal — it does not mean to add their image vectors. This is an example of a "chain".

In other words

$$\Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 \neq \underbrace{(t,0) + (1,t) + (1-t,1) + (0,1-t)}_{=(2,2)}$$

Chains

Definitions

k-cube: $[0,1]^k = I^k$ (where $I = [0,1]$)

singular k-cube in $\mathbb{R}^n$: $\Phi : I^k \to \mathbb{R}^n$

standard k-cube: $\Delta^k : I^k \to \mathbb{R}^k$
$p \mapsto p$

singular k-chain: $C = \sum_s \nu_s \Phi_{(s)}$ ← formal sum

ν_s: real numbers, usually integers

To repeat to point:
If $\Phi_1(t) = (t,1)$ and $\Phi_2(t) = (0,t)$ then $\Phi_1 + 2\Phi_2$ does not equal $(t, 1+2t)$ when we are talking about chains.

integration of a k-form in $\mathbb{R}^n$ along a singular k-chain in $\mathbb{R}^n$: Say $C = \sum_s v_s \Phi_{(s)}$. Then

$$\int_C w = \int_{\sum v_s \Phi_{(s)}} w \overset{\text{def}}{=} \sum v_s \int_{\Phi_{(s)}} w .$$

Example: See the previous example where $C = \varphi_1 + \varphi_2 + \varphi_3 + \varphi_4$, $w = xy\,dx$.

With the same φ_i's and with $\eta = -y\,dx + x\,dy$

$$\int_{2\varphi_1 - \varphi_2} \eta = 2\int_{\varphi_1} \eta - \int_{\varphi_2} \eta = 2\int_{[0,1]} \varphi_1^* \eta - \int_{[0,1]} \varphi_2^* \eta$$

$$= 2\int_{[0,1]} (-0\,dt + t\,d(0)) - \int_{[0,1]} (-t\,d(1) + 1\,dt)$$

$$= -\int_0^1 dt = -1.$$

$\varphi_1(t) = (\underset{x,\,y}{t,0})$, $\varphi_2(t) = (\underset{x,\,y}{1,t})$