Last time $\varphi : D \to \mathbb{R}^n$ w/ $D \subseteq \mathbb{R}^k$,

$$\omega = \sum_I f_I \, dx_I \in \Omega^k \mathbb{R}^n$$

$f_I = f_I(x_1, \dots, x_n)$

$dx_{i_1} \wedge \cdots \wedge dx_{i_k}$

$I = (i_1, \dots, i_k)$

$\varphi^* \omega$ is formed from ω by substituting $x_i = \varphi_i$. i^{th} component of φ

Prop. $\varphi^* \omega = \sum_I f_I \circ \varphi \, \det(J\varphi_I) \, du_I$

where $J\varphi_I$ is the $k \times k$ submatrix formed by all columns of φ and the rows determined by I; and $du_I = du_{i_1} \wedge \cdots \wedge du_{i_k}$.

Pf/ It suffices to consider the case $\omega = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$.

Then $\varphi^* \omega = d\varphi_{i_1} \wedge \cdots \wedge d\varphi_{i_k}$

$$= \left[\sum \frac{\partial \varphi_{i_1}}{\partial u_i} du_i\right] \wedge \cdots \wedge \left[\sum \frac{\partial \varphi_{i_k}}{\partial u_i} du_i\right]$$

$$= \boxed{?}\; du_1 \wedge \cdots \wedge du_k \,.$$

The du_i just serve as "placeholders". We might as well replace them with the standard basis vectors e_i. Our calculation then becomes

$$(\star) \quad \left[\sum \frac{\partial \varphi_{i_1}}{\partial u_i} e_i\right] \wedge \cdots \wedge \left[\sum \frac{\partial \varphi_{i_k}}{\partial u_i} e_i\right]$$

$$= \boxed{?}\; e_1 \wedge \cdots \wedge e_k$$

Where $\boxed{?}$ is the same as above.

But $(\star)$ is just $v_1 \wedge \cdots \wedge v_k$ where $v_j = \left(\frac{\partial \varphi_{i_j}}{\partial u_1}, \ldots, \frac{\partial \varphi_{i_j}}{\partial u_k}\right)$; and we've already seen that $v_1 \wedge \cdots \wedge v_k = \det(v_1, \ldots, v_k)\, e_1 \wedge \cdots \wedge e_k$. $\square$

Remark The Proposition we just proved holds for pulling back k-forms in $\mathbb{R}^n$ along k-surfaces. What about pulling back an l-form for $l < k$? (If $l > 0$, the pullback would be 0.)

Consider the form $\omega = dx_{i_1} \wedge \cdots \wedge dx_{i_l}$. Then

$$\varphi^* \omega = \sum_C g_C \, du_C$$ where each C ranges over all possible $(c_1, \ldots, c_l)$ with $1 \le c_1 < \cdots < c_l \le k$, i.e. $\varphi^* \omega$.

Arguing as in the Proposition, we have g_C is the determinant of the $l \times l$ submatrix of $J\varphi$ consisting of columns C and rows $i_1, \ldots, i_l$.

Basic properties of the pullback and of d

Proposition

Let $\omega \in \Omega^s \mathbb{R}^n$, $\eta \in \Omega^t \mathbb{R}^n$, $\lambda \in \mathbb{R}$, $\varphi : D \xrightarrow{\subseteq \mathbb{R}^k} \mathbb{R}^n$.

1. The pullback and d are linear (over $\mathbb{R}$):

$$\varphi^*(\lambda\omega + \eta) = \lambda\, \varphi^*\omega + \varphi^*\eta,$$

$$d(\lambda\omega + \eta) = \lambda\, d\omega + d\eta.$$

2. $$\varphi^*(\omega \wedge \eta) = \varphi^*\omega \wedge \varphi^*\eta$$

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^s\, \omega \wedge d\eta$$

3. $$\varphi^*(d\omega) = d(\varphi^*\omega)$$

4. If $D \xrightarrow{\varphi} \mathbb{R}^n \xrightarrow{\psi} \mathbb{R}^m$ and $\alpha \in \Omega^l \mathbb{R}^m$, then

$$(\psi \circ \varphi)^*(\alpha) = \varphi^*(\psi^*\alpha).$$

5. $d^2 = 0$.

Discussion / (1) follows directly from the definitions, as does the first part of (2). For the second part of 2, by linearity, it suffices to consider the case $\omega = f\, dx_I$ and $\eta = g\, dx_J$ for some index sets I and J:

$$
\begin{aligned}
d(\omega \wedge \eta) &= d\left((f\, dx_I) \wedge (g\, dx_J)\right) \\
&= d(fg\, dx_I \wedge dx_J) \\
&= [d(fg)] \wedge dx_J \wedge dx_I \\
&= \left[\sum_i \frac{\partial}{\partial x_i}(fg)\, dx_i\right] \wedge dx_I \wedge dx_J \\
&= \left[g \sum_i \frac{\partial f}{\partial x_i} dx_i + f \sum_i \frac{\partial g}{\partial x_i} dx_i\right] \wedge dx_I \wedge dx_J
\end{aligned}
$$

product rule

$$-+-+\cdots \qquad \text{(s jumps)}$$

$$= \left[g \textstyle\sum' \frac{\partial f}{\partial x_i} dx_i\right] \wedge dx_I \wedge dx_J + \left[f \textstyle\sum' \frac{\partial g}{\partial x_i} dx_i\right] \wedge dx_I \wedge dx_J$$

$$= \left[\textstyle\sum' \frac{\partial f}{\partial x_i} dx_i\right] \wedge dx_I \wedge (g\, dx_J) + f\, dx_I \wedge \left[\textstyle\sum' \frac{\partial g}{\partial x_i} dx_i\right] \wedge dx_J$$

$$= (df \wedge dx_i) \wedge (g\, dx_J) + (-1)^s (f\, dx_I) \wedge (dg \wedge dx_J)$$

$$= d\omega \wedge \eta + (-1)\ \omega \wedge d\eta.$$

For (3), by linearity we may assume $\omega = f\, dx_I$. Then

$$\varphi^*(d\omega) = \varphi^*(df \wedge dx_I) = \varphi\Big(\sum_i \frac{\partial f}{\partial x_i} dx_i \wedge dx_I\Big) = \sum_i \frac{\partial f}{\partial x_i}\circ\varphi\ d\varphi_i \wedge d\varphi_I$$

$$= \sum_i \frac{\partial f}{\partial x_i}\circ\varphi \Big(\sum_j \frac{\partial \varphi_i}{\partial u_j} du_j\Big) \wedge d\varphi_I = \sum_{i,j} \frac{\partial f}{\partial x_i}\circ\varphi\ \frac{\partial \varphi_i}{\partial u_j} du_j \wedge d\varphi_I$$

chain rule

$$= \sum' \frac{\partial f\circ\varphi}{\partial u_j} du_j \wedge d\varphi_I = d(f\circ\varphi) \wedge d\varphi_I = d\big(\varphi^*(f \wedge dx_I)\big) = d(\varphi^*\omega).$$

(4) is straightforward: substitute, then substitute again. □

(5): Again by linearity, we may assume $w = f\, dx_I$.

Then $d^2 w = d(dw) = d(df \wedge dx_I) = d\left[\left(\sum_j \frac{\partial f}{\partial x_i} dx_i\right) \wedge dx_I\right]$

$$= \sum_i d\left(\frac{\partial f}{\partial x_i} dx_i \wedge dx_I\right) = \sum_i d\left(\frac{\partial f}{\partial x_i}\right) \wedge dx_i \wedge dx_I$$

$$= \sum_i \left(\sum_j \frac{\partial^2 f}{\partial x_j \partial x_i} dx_j\right) \wedge dx_i \wedge dx_I$$

$$= \sum_{i,j} \frac{\partial^2 f}{\partial x_j \partial x_i} dx_j \wedge dx_i \wedge dx_I$$

For each pair of indices $1 \leq a \leq b \leq n$. There is a term in the sum corresponding to $i = a$ and and $j = b$:

$$\frac{\partial^2 f}{\partial x_b \partial x_a} dx_b \wedge dx_a \wedge dx_I$$

and a term corresponding to $i = b$ and $j = a$:

$$\frac{\partial^2 f}{\partial x_a \partial x_b} \, dx_a \wedge dx_b \wedge dx_I .$$

If $a=b$, the term is 0. Otherwise these two terms cancel out since $dx_b \wedge dx_a = -dx_a \wedge dx_b$.

We are assuming that f is nice enough so that

$$\frac{\partial^2 f}{\partial x_b \partial x_a} = \frac{\partial^2 f}{\partial x_a \partial x_b}, \text{ by the way.}$$

$\square$

Bonus result: If $\omega \in \Omega^s \mathbb{R}^n$ and $\eta \in \Omega^t \mathbb{R}^n$, then

$$\omega \wedge \eta = (-1)^{st} \eta \wedge \omega .$$