Math 212

Example of a pullback

$$\varphi(r,\theta) = (\underset{\substack{\parallel \\ x}}{r\cos\theta},\ \underset{\substack{\parallel \\ y}}{r\sin\theta}) \quad \leftarrow \text{ 2-surface in } \mathbb{R}^2$$

$$\omega = (x^2+y^2)^{\frac{3}{2}}\, dx \wedge dy \in \Omega^2 \mathbb{R}^2 \quad \leftarrow \text{ 2-form in } \mathbb{R}^2$$

To calculate the pullback: substitute $x = r\cos\theta$, $y = r\sin\theta$;

$$\varphi^*\omega = \left((r\cos\theta)^2 + (r\sin\theta)^2\right)^{\frac{3}{2}} d(r\cos\theta) \wedge d(r\sin\theta)$$

$$= (r^2)^{\frac{3}{2}} (\cos\theta\, dr - r\sin\theta\, d\theta) \wedge (\sin\theta\, dr + r\cos\theta\, d\theta)$$

$$= r^3 \left[r\cos^2\theta + r\sin^2\theta\right] dr \wedge d\theta$$

$$= r^3 \cdot r \; dr \wedge d\theta$$

$\star$ stretching factor!

$$= r^4\, dr \wedge d\theta$$

Formal definition of pullback

Def. If $\varphi : D \to \mathbb{R}^n$ is a k-surface in $\mathbb{R}^n$ and $\omega \in \Omega^\ell \mathbb{R}^n$, define the pullback of ω along φ, denoted $\varphi^* \omega \in \Omega^\ell D$ as follows:

$$\omega = \sum_I f_I \, dx_I$$ where the sum is over $I = (i_1, \ldots, i_\ell)$,

$f_I = f_I(x_1, \ldots, x_n)$, and $dx_I := dx_{i_1} \wedge \cdots \wedge dx_{i_\ell}$.

Then $$\varphi^* \omega := \sum_I f \circ \varphi \; d\varphi_I$$ ← This is notation for

$$:= \sum_I f(\varphi_1(u), \ldots, \varphi_n(u)) \, d\varphi_{i_1} \wedge \cdots \wedge d\varphi_{i_\ell}$$

In other words, set $x_i = \varphi_i$, the i^{th} component of φ, and substitute into ω.

Special case: If φ is a 0-surface, then

$\varphi : \mathbb{R}^0 \to \mathbb{R}^n$. Say $\varphi(()) = p \in \mathbb{R}^n$. If $\omega \in \Omega^k \mathbb{R}^n$, then

$\varphi^* \omega = 0$ unless $k = 0$, in which case $\omega = f(x_1, \dots, x_n)$, a function on $\mathbb{R}^n$. In the latter case, $\varphi^* \omega = \varphi^* f := f(p)$. So the pullback of a 0-form is just evaluation.

Example $\varphi : \mathbb{R}^0 \to \mathbb{R}^3$.

$$() \mapsto (1, 3, -2)$$

$$\varphi^*(x\,dx \wedge dy + dy \wedge dz) = 1 \cdot d(1) \wedge d(3) + d(3) \wedge d(-2) = 0$$

and $\varphi^*(\underbrace{x^2 + y^2}_{\text{0-form}}) = 1^2 + 3^2 = 10.$

Integrating a k-form along a k-surface

If $\varphi : D \to \mathbb{R}^n$ w/ $D \subseteq \mathbb{R}^k$, and $\omega \in \Omega^k \mathbb{R}^n$,

define

$$\int_\varphi \omega = \int_D \varphi^* \omega .$$

Special case: For a constant $\lambda \in \mathbb{R}$, define $\int_{\mathbb{R}^0} \lambda = \lambda$. Hence, if φ is a 0-surface with $\varphi(()) = p \in \mathbb{R}^n$ and f is a function on $\mathbb{R}^n$, then $\int_\varphi f = \int_{\mathbb{R}^0} \varphi^* f$
$= \int_{\mathbb{R}^0} f(p) = f(p)$.

Example

$$\varphi : [0, 2\pi] \to \mathbb{R}^2, \quad \omega = y\,dx - x\,dy \in \Omega^1 \mathbb{R}^2$$
$$t \mapsto (\cos t, \sin t)$$

$$\int_\varphi \omega = \int_\varphi y\,dx - x\,dy = \int_{[0,2\pi]} \sin t \, d(\cos t) - \cos(t)\, d(\sin t)$$

$$= \int_{[0,2\pi]} (-\sin^2 t - \cos^2 t)\, dt = \int_{[0,2\pi]} -1 \, dt = -2\pi$$

Example

$$\varphi : [0,1]^2 \longrightarrow \mathbb{R}^3 \quad , \quad \omega = dx \wedge dy + y\, dx \wedge dz$$
$$(u,v) \longmapsto (u, v, u^2+v^2)$$

$$\int_\varphi \omega = \int_{[0,1]^2} \varphi^* \omega = \int_{[0,1]^2} du \wedge dv + v\, du \wedge d(u^2+v^2)$$

$$= \int_{[0,1]^2} du \wedge dv + v\, du \wedge (2u\, du + 2v\, dv)$$

$$= \int_{[0,1]^2} du \wedge dv + 2v^2\, du \wedge dv = \int_{[0,1]^2} (1+2v^2)\, du \wedge dv$$

def. of integral of a 2-form over a region in $\mathbb{R}^2$

$$= \int_{[0,1]^2} (1+2v^2) = \int_0^1 \int_0^1 1+2v^2\, du\, dv = 1 + \frac{2}{3} = \frac{5}{3}.$$

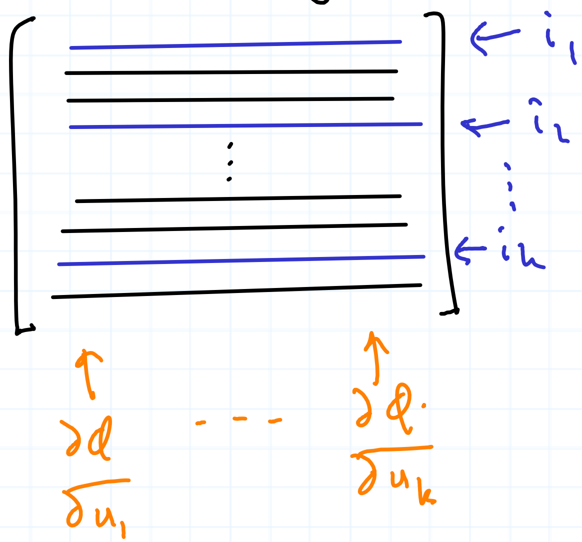
Example $\varphi : () \to \mathbb{R}^4$, $\omega = x^2+y^2+z^2+t^2$. $\int_\varphi \omega = \int_{\mathbb{R}^0} 3^2+0^2+2^2+(-1)^2 = 14.$

$\varphi(()) = (3,0,2,-1)$

coords. $u_1,\dots,u_k$

Prop. Let $\varphi : D \to \mathbb{R}^n$ be a k-surface in $\mathbb{R}^n$ and let $1 \le i_1 < i_2 < \cdots < i_k \le n$. Let $I = (i_1, \dots, i_n)$

So $dx_I \overset{\text{def.}}{=} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, and define $J\varphi_I$ to be the $k \times k$ submatrix of the Jacobian of φ formed by taking all k columns but only rows $i_1, \dots, i_k$:

$J\varphi_I =$ 

Then $\varphi^* dx_I = \det J\varphi_I \; du_{i_1} \wedge \cdots \wedge du_{i_k}$.

Example

$$\varphi(u,v) = (\cos u,\ \sin v,\ uv,\ 2u+3v) \in \mathbb{R}^4$$

coordinates $x_1, \dots, x_4$

$$J\varphi = \begin{array}{cc} \begin{bmatrix} -\sin u & 0 \\ 0 & \cos v \\ v & u \\ 2 & 3 \end{bmatrix} & \begin{array}{c} dx_1 \\ dx_2 \\ dx_3 \\ dx_4 \end{array} \end{array}$$

$$I = (2,4), \quad dx_I = dx_2 \wedge dx_4, \quad J\varphi_I = \begin{bmatrix} 0 & \cos v \\ 2 & 3 \end{bmatrix}$$

By the def. of the pullback

$$\varphi^* dx_I = \varphi^*(dx_2 \wedge dx_4) = d(\sin v) \wedge d(2u+3v)$$

$$= (\cos v\, dv) \wedge (2du + 3dv)$$

$$= -2\cos v\, du \wedge dv$$

By the Proposition:

$$\varphi^* dx_I = \det J\varphi_I \, du \wedge dv = \det \begin{bmatrix} 0 & \cos v \\ 2 & 3 \end{bmatrix} du \wedge dv$$

$$= -2\cos v \, du \wedge dv. \quad \checkmark$$

Example With $\Phi(u,v) = (\cos u, \sin v, uv, 2u+3v) \in \mathbb{R}^4$, as above but with $\omega = x_3\, dx_1 \wedge dx_3 + x_1 x_2\, dx_2 \wedge dx_3$, compute $\Phi^* \omega$.

Solution / The Jacobian is

$$J\Phi = \begin{array}{cc} \begin{bmatrix} -\sin u & 0 \\ 0 & \cos v \\ v & u \\ 2 & 3 \end{bmatrix} & \begin{array}{c} dx_1 \\ dx_2 \\ dx_3 \\ dx_4 \end{array} \end{array}.$$

So

$$\Phi^* \omega = uv \det \begin{bmatrix} -\sin u & 0 \\ v & u \end{bmatrix} du \wedge dv + \cos u \sin v \det \begin{bmatrix} 0 & \cos v \\ v & u \end{bmatrix} du \wedge dv$$

$$= -u^2 v \sin u \, du \wedge dv - v \cos u \cos v \sin v \, du \wedge dv$$

$$= \left[-u^2 v \sin u - v \cos u \cos v \sin v \right] du \wedge dv .$$