* Go over HW

Math 212

☆ No quiz this week.

Integrating an n-form over $D \subseteq \mathbb{R}^n$

$\omega \in \Omega^n \mathbb{R}^n$ must have the form $f\, dx_1 \wedge \cdots \wedge dx_n$.

Define

Usual integral

$$\int_D \omega = \int_D f\, dx_1 \wedge \cdots \wedge dx_n := \int_D f\, dx_1 \cdots dx_n$$

Important: The variables in the wedge part $dx_1 \wedge \cdots \wedge dx_n$ must have the same orientation as $\mathbb{R}^n$ itself. Once the wedges are dropped, Fubini says order does not matter. For example,

$$\int_{[0,1]^2} f(x,y)\, dx \wedge dy = \int_0^1 \int_0^1 f(x,y)\, dx\, dy = \int_0^1 \int_0^1 f(x,y)\, dy\, dx$$

$$= -\int_{[0,1]^2} f(x,y)\, dy \wedge dx$$

Note.

Example $\omega = (x - xy)\, dy \wedge dx \in \Omega^2 \mathbb{R}^2$

(coords. on $\mathbb{R}^2$ ordered x, y, as usual)

$$\int_{[0,1]^2} \omega = \int_{[0,1]^2} (x - xy)\, dy \wedge dx$$

$$= -\int_{[0,1]^2} (x - xy)\, dx \wedge dy$$

$$= -\int_{[0,1]^2} (x - xy)\, dx\, dy$$

$$= -\int_0^1 \int_0^1 (x - xy)\, dx\, dy$$

$$= -\int_0^1 \left(\tfrac{1}{2} - \tfrac{1}{2} y \right) dy$$

$$= -\left[\tfrac{1}{2} - \tfrac{1}{4} \right] = -\tfrac{1}{2}.$$

k-surfaces in $\mathbb{R}^n$

A k-surface in $\mathbb{R}^n$ is a function

$$\phi : D \longrightarrow \mathbb{R}^n$$

where $D \subseteq \mathbb{R}^k$.

Examples (with unspecified domains)

* A 2-surface in $\mathbb{R}^4$: $\phi(u,v) = (u, v, u\cos(v), u\sin(v))$
* A 0-surface in $\mathbb{R}^3$: $\phi(()) = (1, 3, -7)$

Note: $\mathbb{R}^0$ has a single point, $()$.

* A 1-surface in $\mathbb{R}^5$: $\phi(t) = (t, t^2, t^3, t^4, t^5)$ (a parametrized curve)
* A 2-surface in $\mathbb{R}^2$: $\phi(r,\theta) = (r\cos\theta, r\sin\theta)$

Pullbacks by example

I. $\varphi : [0,1]^2 \longrightarrow \mathbb{R}^3$, $\omega = y\,dx - x\,dy + z^2\,dz \in \Omega^1\mathbb{R}^3$

$$(u,v) \longmapsto (\underset{x}{\underset{\shortparallel}{u}}, \underset{y}{\underset{\shortparallel}{v}}, \underset{z}{\underset{\shortparallel}{uv}})$$

The pullback of ω along φ :

$$\varphi^*(\omega) = v\,d(u) - u\,d(v) + (uv)^2 d(uv)$$

$$= v\,du - u\,dv + (uv)^2 [v\,du + u\,dv]$$

$$= (v + u^2v^3)\,du + (-u + u^3v^2)\,dv \in \Omega^1\mathbb{R}^2$$

(or $\Omega^1[0,1]^2$, more precisely)

II. $\varphi : [0,1]^2 \longrightarrow \mathbb{R}^3$

$$(u,v) \longmapsto (\underset{x}{\underset{\shortparallel}{u+v}}, \underset{y}{\underset{\shortparallel}{3v}}, \underset{z}{\underset{\shortparallel}{u\cos(v)}})$$

$$\omega = x\,dx \wedge dy + y\,dy \wedge dz$$

5

$$\varphi^* \omega = (u+v)\, d(u+v) \wedge d(3v) + 3v\, d(3v) \wedge d(u\cos(v))$$

$$= (u+v)\left[(du+dv)\wedge(3dv)\right] + 3v\left[(3dv) \wedge (\cos(v)\,du - u\sin(v)\,dv)\right]$$

$$= (u+v)\left[3\, du \wedge dv\right] + 3v\left[-3\cos(v)\, du \wedge dv\right]$$

$$= \left[3(u+v) - 9v\cos(v)\right] du \wedge dv \in \Omega^2 \mathbb{R}^2 \quad \left(\text{really } \Omega^2 [0,1]^2\right)$$