Differential forms

Math 212

Monday, 3/4/13

$\Omega^k \mathbb{R}^n$: differential k-forms on $\mathbb{R}^n$

- alternating tensors
- on the base set of symbols $dx_1, \ldots, dx_n$
- with $\mathbb{R}$-valued functions as coefficients

Examples

2-form on $\mathbb{R}^3$: $x^2\, dx \wedge dy - y\, dy \wedge dz$

1-form on $\mathbb{R}^4$: $dx + e^t\, dy + (3x^2 - y)\, dz - dt$

0-form on $\mathbb{R}^2$: $x^2 + xy$

$\Omega^0 \mathbb{R}^n = \mathbb{R}$-valued functions on $\mathbb{R}^n$

The algebra of k-forms

$\uparrow$

(multiplication)

Let $\Omega^\bullet \mathbb{R}^n = \oplus_{k \geq 0} \Omega^k \mathbb{R}^n$ denote finite linear combinations of k-forms with k varying.

Example

An element of $\Omega^\bullet \mathbb{R}^4$:

$$3x + (y + 4t)\, dx \wedge dz + e^{yt} dx \wedge dy \wedge dz \wedge dt.$$

The algebra of k-forms

Ring structure

If $\omega \in \Omega^k\mathbb{R}^n$ and $\eta \in \Omega^\ell\mathbb{R}^n$, define their product to be $\omega \wedge \eta \in \Omega^{k+\ell}\mathbb{R}^n$.

Express the following in standard (alphabetical) form:

1. $(t\,ds + s\,dt) \wedge (s\,ds + t\,dt) = ?$ $\quad (t^2 - s^2)\, ds \wedge dt$
2. $(3x + y\,dx \wedge dy + z\,dy \wedge dz) \wedge (dz \wedge dt) = ?$ $\quad 3x\, dz \wedge dt + y\, dx \wedge dy \wedge dz \wedge dt$
3. $(2y\,dx + 3z\,dy) \wedge (-4\,dx + z\,dy) = ?$ $\quad (2yz + 12z)\, dx \wedge dy$

The exterior derivative

The exterior derivative is the function

$$d\colon \Omega^k\mathbb{R}^n \to \Omega^{k+1}\mathbb{R}^n,$$

defined as follows.

- For $k = 0$, let $f \in \Omega^0\mathbb{R}^n$, i.e., a function on $\mathbb{R}^n$. Then

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x)\, dx_i.$$

- For $k > 0$, define

$$d(f\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = df \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Examples

1. $d(x^2 + y) = ?$ $\quad 2x\,dx + dy$
2. $d(xy\,dx) = ?$ $\quad (y\,dx + x\,dy)\wedge dx = -x\,dx\wedge dy$
3. $d(xy\,dx \wedge dy) = ?$ $\quad d(xy)\wedge dx\wedge dy = (y\,dx + x\,dy)\wedge dx\wedge dy = 0$
4. $d((s+t)ds + (st^2)\,dt) = ?$ $\quad d(s+t)\wedge ds + d(st^2)\wedge dt$
   $= (ds + dt)\wedge ds + (t^2 ds + 2st\,dt)\wedge dt$
   $= dt\wedge ds + t^2 ds\wedge dt = (t^2-1)\,ds\wedge dt$
5. $d(\cos(uv))$
   $= -v\sin(uv)\,du - u\sin(uv)\,dv$
6. $d^2(x^2 + y) = ?$
   $= d(d(x^2+y)) = d(2x\,dx + dy) = d(2x)\wedge dx + d(1)\wedge dy$
   $= 2dx\wedge dx + 0\wedge dy = 0.$

Integration of n-forms on $\mathbb{R}^n$

For $D \subset \mathbb{R}^n$ and $\omega \in \Omega^n \mathbb{R}^n$, we must have

$$\omega = f\, dx_1 \wedge \cdots \wedge dx_n, \qquad f \colon \mathbb{R}^n \to \mathbb{R}.$$

Definition:

$$\int_D f\, dx_1 \wedge \cdots \wedge dx_n = \int_D f.$$

Note: Order (orientation) is important:

$$f\, dx_1 \wedge \cdots \wedge dx_n \neq f\, dx_2 \wedge dx_1 \wedge dx_3 \wedge \cdots \wedge dx_n.$$