

Differential forms

Math 212

Friday, 3/1/13

Alternating tensors

Mod out k -tensors on $\mathbb{R}^n$ by the further relation:

$$v_1 \otimes \cdots \otimes v_k = 0 \quad \text{if } v_i = v_j \text{ for some } i \neq j.$$

The equivalence class of $v_1 \otimes v_2 \cdots \otimes v_k$ is denoted

$$v_1 \wedge v_2 \wedge \cdots \wedge v_k.$$

This defines $\Lambda^k \mathbb{R}^n$, the space of *alternating k -tensors on $\mathbb{R}^n$* .

Example:

$$3e_1 \wedge (4e_1 + 2e_2) = ?$$

$$= 12e_1 \wedge e_1 + 6e_1 \wedge e_2 = 6e_1 \wedge e_2$$

Alternating tensor is alternating

Proposition

$$\begin{aligned} v_1 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_j \wedge \cdots \wedge v_k = \\ -v_1 \wedge \cdots \wedge v_j \wedge \cdots \wedge v_i \wedge \cdots \wedge v_k. \end{aligned}$$

for $i \neq j$.

For example,

$$\begin{aligned} 0 &= u \wedge (v + w) \wedge p \wedge (v + w) = ? \\ &= u \wedge v \wedge p \wedge (v + w) + u \wedge w \wedge p \wedge (v + w) \\ &= u \wedge v \wedge p \wedge v + u \wedge v \wedge p \wedge w + u \wedge w \wedge p \wedge v + u \wedge w \wedge p \wedge w \\ &= 0 + u \wedge v \wedge p \wedge w + u \wedge w \wedge p \wedge v \\ &\Rightarrow u \wedge \underline{v} \wedge p \wedge \underline{w} = -u \wedge \underline{w} \wedge p \wedge \underline{v}. \end{aligned}$$

Compelling examples

$$e_1 = (1, 0)$$

$$e_2 = (0, 1)$$

For $(a, b), (c, d) \in \mathbb{R}^2$,

$$(a, b) \wedge (c, d) = \boxed{?} e_1 \wedge e_2.$$

$= (ae_1 + be_2) \wedge (ce_1 + de_2)$
 $= etc. = (ad - bc)e_1 \wedge e_2$

For $v_1, \dots, v_n \in \mathbb{R}^n$,

$$v_1 \wedge \dots \wedge v_n = \boxed{?} e_1 \wedge \dots \wedge e_n.$$

$\curvearrowright \det(v_1, \dots, v_n)$

Point:

$$(v_1, \dots, v_n) \rightarrow \boxed{\text{blah}} e_1 \wedge \dots \wedge e_n.$$

is a multilinear, alternating function taking the value 1 when $v_i = e_i$ for all i . Therefore, the box must be filled in by $\det(v_1, \dots, v_n)$.

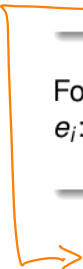
Compelling examples

Let $(a, b, c), (d, e, f) \in \mathbb{R}^3$. Write the following in terms of the standard basis vectors e_j :

$$(a, b, c) \wedge (d, e, f) = ?$$

For $v_1, \dots, v_{n-1} \in \mathbb{R}^n$. Again, write the following in terms of the e_j :

$$v_1 \wedge \dots \wedge v_{n-1}.$$


$$= (ae_1 + be_2 + ce_3) \wedge (de_1 + ee_2 + fe_3)$$

$$= (ae - bd)e_1 \wedge e_2 + (af - cd)e_1 \wedge e_3 + (bf - ce)e_2 \wedge e_3$$

(compare this to the cross product, $(a, b, c) \times (d, e, f)$)

A new twist

We've defined k -tensors on $\mathbb{R}^n$, e.g.

$$(3e_1 + 2e_2) \otimes (2e_2 - 5e_3) = 6e_1 \otimes e_2 - 15e_1 \otimes e_3 - 10e_2 \otimes e_3.$$

Now, instead of using $\mathbb{R}$ for scalars, use functions $f: \mathbb{R}^n \rightarrow \mathbb{R}$:

$$\begin{aligned} \tau &= ((3x^2 + y)e_1 + e^{x^2+z^2}e_2) \otimes (ze_2 + xy e_3) = ? \\ &= (3x^2+y)ze_1 \otimes e_2 + (3x^2+y)xy e_1 \otimes e_3 + ze^{x^2+z^2}e_2 \otimes e_2 \\ &\quad + xy e^{x^2+z^2}e_2 \otimes e_3 \\ &\quad \tau(1,2,3) = ? \\ &= 10e_1 \otimes e_2 + 15e_1 \otimes e_3 + 3e^{10}e_2 \otimes e_3 + 2e^{10}e_2 \otimes e_3. \end{aligned}$$

Differential forms

Consider the set of symbols

$$S = \{dx_1, \dots, dx_n\}.$$

Form the free vector space (module, really, but never mind), mod out by stuff to get the k -tensors on these symbols (with $\mathbb{R}$ -valued functions as scalars), mod out again to get alternating tensors on these symbols. The result is the space:

$$\Omega^k \mathbb{R}^n = \text{differential } k\text{-forms on } \mathbb{R}^n.$$

A 2-form on $\mathbb{R}^3$: $\omega = e^{xz} dx \wedge dy - (x^2 - 2y)dy \wedge dz$.

$$\omega(0, 1, 1) = ?$$

$$= dx \wedge dy + 2 dy \wedge dz$$