Differential forms

Math 212

Friday, 3/1/13

Alternating tensors

Mod out k-tensors on $\mathbb{R}^n$ by the further relation:

$$v_1 \otimes \cdots \otimes v_k = 0 \quad \text{if } v_i = v_j \text{ for some } i \neq j.$$

The equivalence class of $v_1 \otimes v_2 \cdots \otimes v_k$ is denoted

$$v_1 \wedge v_2 \wedge \cdots \wedge v_k.$$

This defines $\Lambda^k \mathbb{R}^n$, the space of *alternating k-tensors on* $\mathbb{R}^n$.

Example:

$$3e_1 \wedge (4e_1 + 2e_2) = ?$$

$$= 12 e_1 \wedge e_1 + 6 e_1 \wedge e_2 = 6 e_1 \wedge e_2$$

Alternating tensor is alternating

Proposition

$$v_1 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_j \wedge \cdots \wedge v_k = \\ -v_1 \wedge \cdots \wedge v_j \wedge \cdots \wedge v_i \wedge \cdots \wedge v_k.$$

for $i \neq j$.

For example,

$$0 = u \wedge (v + w) \wedge p \wedge (v + w) = ?$$

$= u \wedge v \wedge p \wedge (v+w) + u \wedge w \wedge p \wedge (v+w)$

$= u \wedge v \wedge p \wedge v + u \wedge v \wedge p \wedge w + u \wedge w \wedge p \wedge v + u \wedge w \wedge p \wedge w$

$= 0 + u \wedge v \wedge p \wedge w + u \wedge w \wedge p \wedge v$

$\Longrightarrow u \wedge \underline{v} \wedge p \wedge \underline{w} = - u \wedge \underline{w} \wedge p \wedge \underline{v}$.

Compelling examples

$e_1 = (1,0)$
$e_2 = (0,1)$

For $(a, b), (c, d) \in \mathbb{R}^2$,

$$(a, b) \wedge (c, d) = \boxed{?}\, e_1 \wedge e_2.$$

$= (ae_1 + be_2) \wedge (ce_1 + de_2)$
$= \text{etc.} = (ad - bc) e_1 \wedge e_2$

For $v_1, \ldots, v_n \in \mathbb{R}^n$,

$$v_1 \wedge \cdots \wedge v_n = \boxed{?}\, e_1 \wedge \cdots \wedge e_n.$$

$\det(v_1, \ldots, v_n)$

Point:

$$(v_1, \ldots, v_n) \to \boxed{\text{blah}}\, e_1 \wedge \cdots \wedge e_n.$$

is a multilinear, alternating function taking the value 1 when $v_i = e_i$ for all i. Therefore, the box must be filled in by $\det(v_1, \ldots, v_n)$.

Compelling examples

Let $(a,b,c),(d,e,f) \in \mathbb{R}^3$. Write the following in terms of the standard basis vectors e_i:

$$(a,b,c) \wedge (d,e,f) = ?$$

For $v_1, \ldots, v_{n-1} \in \mathbb{R}^n$. Again, write the following in terms of the e_i:

$$v_1 \wedge \cdots \wedge v_{n-1}.$$

$$= (ae_1 + be_2 + ce_3) \wedge (de_1 + ee_2 + fe_3)$$

$$= (ae - bd)\, e_1 \wedge e_2 + (af - cd)\, e_1 \wedge e_3 + (bf - ce)\, e_2 \wedge e_3$$

(compare this to the cross product, $(a,b,c) \times (d,e,f)$)

A new twist

We've defined k-tensors on $\mathbb{R}^n$, e.g.

$$(3e_1 + 2e_2) \otimes (2e_2 - 5e_3) = 6e_1 \otimes e_2 - 15e_1 \otimes e_3 - 10e_2 \otimes e_3.$$

Now, instead of using $\mathbb{R}$ for scalars, use functions $f \colon \mathbb{R}^n \to \mathbb{R}$:

$$\tau = ((3x^2 + y)e_1 + e^{x^2+z^2} e_2) \otimes (z\, e_2 + xy\, e_3) = ?$$

$$= (3x^2+y)z\, e_1 \otimes e_2 + (3x^2+y)xy\, e_1 \otimes e_3 + z e^{x^2+z^2} e_2 \otimes e_2 + xy e^{x^2+z^2} e_2 \otimes e_3$$

$$\tau(1, 2, 3) = ?$$

$$= 10 e_1 \otimes e_2 + 15 e_1 \otimes e_3 + 3e^{10} e_2 \otimes e_3 + 2e^{10} e_2 \otimes e_3.$$

Differential forms

Consider the set of symbols

$$S = \{dx_1, \ldots, dx_n\}.$$

Form the free vector space (module, really, but never mind), mod out by stuff to get the k-tensors on these symbols (with $\mathbb{R}$-valued functions as scalars), mod out again to get alternating tensors on these symbols. The result is the space:

$$\Omega^k \mathbb{R}^n = \text{differential } k\text{-forms on } \mathbb{R}^n.$$

A 2-form on $\mathbb{R}^3$: $\omega = e^{xz}\, dx \wedge dy - (x^2 - 2y) dy \wedge dz$.

$$\omega(0, 1, 1) = ?$$

$$= dx \wedge dy + 2\, dy \wedge dz$$