Math 212

Quiz.

1. (a) State the spherical change of coordinates.
   (b) Draw a picture to show what it means.
   (c) What is the stretching factor $(|\det J\varphi|)$?

2. Integrate $e^{(x^2+y^2+z^2)^{\frac{3}{2}}}$ $\left(= \exp\left((x^2+y^2+z^2)^{\frac{3}{2}}\right)\right)$ over the $\frac{1}{8}$ th of the unit ball in the positive orthant $\left(\{(x,y,z) \in \mathbb{R}^3 : x,y,z \geq 0,\ x^2+y^2+z^2 \leq 1\}\right)$.

Tensors

Let S be any set. The free vector space (over $\mathbb{R}$), FS, on S is the collection of finite formal sums: $\sum_{s \in S} a_s s$ with $a_s \in \mathbb{R}$

and all but a finite number of the a_s equal to 0, with the natural linear structure:

$$\sum a_s s + \sum b_s s = \sum (a_s + b_s) s$$

$$\lambda \sum a_s s = \sum (\lambda a_s) s \qquad \text{for } \lambda \in \mathbb{R}.$$

Note: $0 = \sum_{s \in S} 0 \cdot s \in FS$. Also S is a basis for FS.

Example. $S = \{a, b, c, d\}$

$$3(3a + \pi b - d) + 2(b + 4d) = 9a + (3\pi + 2) b + 5d$$

The example we are interested in is $S = \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{k\text{-times}}$.

We'll call this $F^k \mathbb{R}^n$.

Example $5\big((3,2,4),(1,0,5)\big) - 3\big((8,-5,2),(4,6,12)\big) \in F^2 \mathbb{R}^3$.

Example. $2\big((3,1,3),(1,1,2)\big) \neq \big((6,2,6),(2,2,4)\big)$ in $F^2\mathbb{R}^3$.

We now "mod out" by the relations:

(1) $a(v_1,\dots,v_k) = (av_1, v_2, \dots, v_k) = (v_1, av_2, v_3, \dots, v_k) = \dots = (a_1, \dots, v_{k-1}, av_k)$.

(2) $(v_1,\dots,v_{i-1}, v+w, v_{i+1},\dots,v_k) = (v_1,\dots,v_{i-1},v,v_{i+1},\dots,v_k) + (v_1,\dots,v_{i-1},w,v_{i+1},\dots,v_k)$

for each i.

a vector space over $\mathbb{R}$

to get the k-fold tensor space, $\underbrace{\mathbb{R}^n \otimes \dots \otimes \mathbb{R}^n}_{k\text{-times}} = \bigotimes_{i=1}^{k} \mathbb{R}^n = (\mathbb{R}^n)^{\otimes k} = T^k\mathbb{R}^n$

All notation for the same thing.

The equivalence class of $(\underbrace{v_1,\dots,v_k}_{v_i \in \mathbb{R}^n,\ i=1,\dots,k}) \in T^k\mathbb{R}^n$ is denoted $v_1 \otimes \dots \otimes v_k$.

Example For $u, v, w \in \mathbb{R}^3$, we have $3(u \otimes v) = (3u) \otimes v = u \otimes 3v$

and $(5u + v) \otimes 6w = 5u \otimes 6w + v \otimes 6w = 30\, u \otimes w + 6\, v \otimes w$.

If $u = (1,2,3)$ and $v = (3,2,1)$, we have

$$2(u \otimes v) = (2u) \otimes v = (2,4,6) \otimes (3,2,1).$$

In terms of the standard basis vectors,

$$\begin{aligned}(2,4,6) \otimes (3,2,1) &= (2e_1 + 4e_2 + 6e_3) \otimes (3e_1 + 2e_2 + e_3) \\ &= 6\, e_1 \otimes e_1 + 4\, e_1 \otimes e_2 + 2\, e_1 \otimes e_3 \\ &\quad + 12\, e_2 \otimes e_1 + 8\, e_2 \otimes e_2 + 4\, e_2 \otimes e_3 \\ &\quad + 18\, e_3 \otimes e_1 + 12\, e_3 \otimes e_2 + 6\, e_3 \otimes e_3.\end{aligned}$$

Note: $e_1 \otimes e_2 \neq e_2 \otimes e_1$.

Prop. $v_1 \otimes \cdots \otimes v_k = 0$ if $v_i = 0$ for some i.

Pf/ For ease of notation, suppose $v_1 = 0 \in \mathbb{R}^n$. Then

0 times any vector is the zero vector

$$\vec{0} = 0\,(0 \otimes v_2 \otimes \cdots \otimes v_k) = (0 \cdot 0) \otimes v_2 \otimes \cdots \otimes v_k = 0 \otimes v_2 \otimes \cdots \otimes v_k. \qquad \square$$

The zero vector in $T^k\mathbb{R}^n$

We can bring scalars "inside" with tensor products.