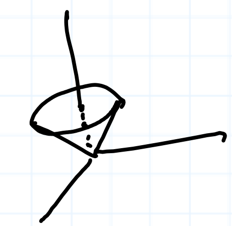
Start w/ #4

Math 212 Practice problems

1. $\int_S \sqrt{x^2+y^2}$, S = region bounded by $z^2 = x^2+y^2$ and $z = 1$



Solution / Use cylindrical coordinates, $x = r\cos\theta$, $y = r\sin\theta$:

$$\int_S \sqrt{x^2+y^2} = \int_0^1 \int_0^{2\pi} \int_0^z \sqrt{r^2\cos^2\theta + r^2\sin^2\theta}\; \underbrace{r}_{\text{stretching factor}}\, dr\, d\theta\, dz = \int_0^1 \int_0^{2\pi} \int_0^z r^2\, dr\, d\theta\, dz$$

$$= \int_0^1 \int_0^{2\pi} \frac{1}{3} z^3\, d\theta\, dz = \frac{2\pi}{3} \int_0^1 z^3\, dz = \frac{\pi}{6}$$

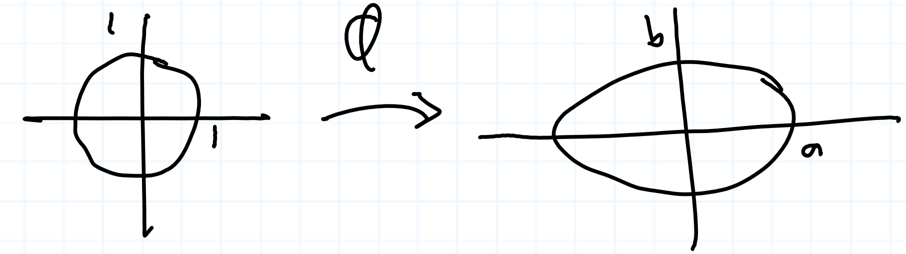
2. Area inside an ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ [Hint: change coordinates $\bigcirc \rightarrow ⬭ \xrightarrow{\int} \mathbb{R}$]

Solution / Let $\Phi(x,y) = (ax, by)$. If (x,y) satisfies $x^2+y^2 = 1$, then $\tilde{x} = ax$, $\tilde{y} = by$ satisfies $\left(\frac{\tilde{x}}{a}\right)^2 + \left(\frac{\tilde{y}}{b}\right)^2 = 1$, i.e. Φ maps the unit circle to an ellipse. Since

$$|\det J\Phi| = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab, \qquad (\text{assuming } a, b > 0)$$

we have

$$\text{area} = \int_{\bigcirc} 1 = \int_{\bigcirc} ab = \pi ab.$$

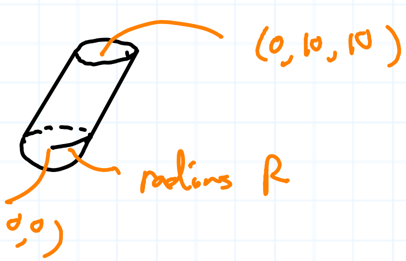


3. Volume of an ellipsoid $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=1$.

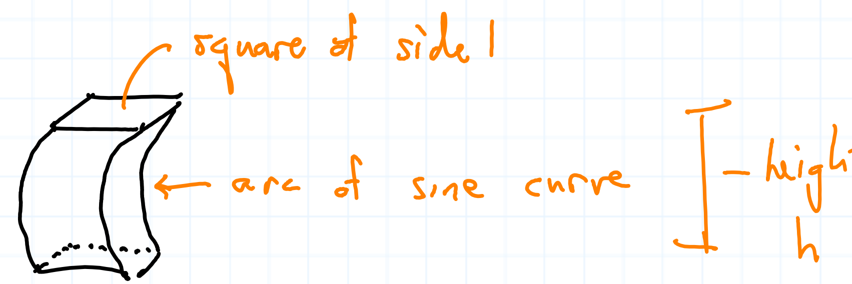
Solution / Using the change of coordinates $\Phi(x,y,z)=(ax,by,cz)$, and proceeding as above, we get: $\text{volume}(\ominus) = abc\,(\text{volume of unit sphere}) = \frac{4}{3}\pi abc$.

4

4. Volume of: (a)



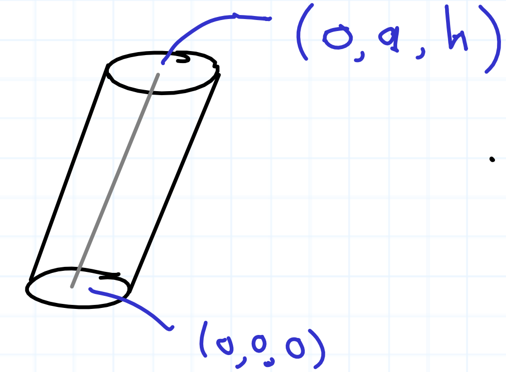
(b)



Solution / (a) By Fubini, $\text{vol}(\square) = \int_{z=0}^{10} \underbrace{\int_y\int_x \chi_B}_{\text{cross-sectional area at height } z} = \int_{z=0}^{10} \pi R^2 = 10\pi R^2.$

(b) Similarly, $\text{vol}(\square) = h.$

To double check, let's parametrize



$$\Phi(t, r, \theta) = t\,(0, a, h) + (r\cos\theta, r\sin\theta, 0)$$

$$= (r\cos\theta,\ at + r\sin\theta,\ th).$$

$$0 \le t \le 1, \quad 0 \le r \le R, \quad 0 \le \theta \le 2\pi$$

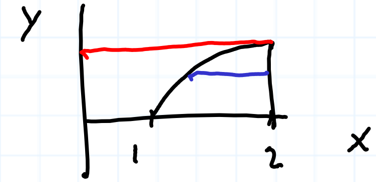
$$J\Phi = \begin{bmatrix} 0 & \cos\theta & -r\sin\theta \\ a & \sin\theta & r\cos\theta \\ h & 0 & 0 \end{bmatrix}, \quad \det J\Phi = h \det \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$$= hr.$$

$$\mathrm{vol}(\ ) = \int_0^{2\pi}\int_0^{R}\int_0^{1} hr\,dt\,dr\,d\theta = \int_0^{2\pi}\int_0^{R} hr\,dr\,d\theta = \int_0^{2\pi} \frac{hR^2}{2}\,d\theta = \pi R^2 h.$$

(picture w/ Sage for various values of a)

5. $\int_1^2 \int_0^{\ln x} (x-1)\sqrt{1+e^{2y}}\, dy\, dx$.



$y = \ln x \Rightarrow x = e^y$

Solution / Change the order of integration:

$$\int_0^{\ln 2}\int_{e^y}^{2} (x-1)\sqrt{1+e^{2y}}\, dx\, dy = \int_0^{\ln 2} \left[\left(\frac{x^2}{2} - x\right)\sqrt{1+e^{2y}} \Big|_{e^y}^{2} \right] dy$$

$$= \int_0^{\ln 2} \left(e^y - \tfrac{1}{2}e^{2y}\right)\sqrt{1+e^{2y}}\, dy = -\frac{1}{2}\int_0^{\ln 2} e^{2y}(1+e^{2y})^{\frac{1}{2}}\, dy + \int_0^{\ln 2} \underbrace{e^y\sqrt{1+e^{2y}}\, dy}$$

$u = e^y \quad du = e^y\, dy \qquad u(\ln 2) = e^{\ln 2} = 2, \quad u(0) = e^0 = 1$

$$= -\frac{1}{2}\left[\frac{1}{3}(1+e^{2y})^{\frac{3}{2}} \Big|_0^{\ln 2} \right] + \int_1^2 \sqrt{1+u^2}\, du$$

$$= -\frac{1}{6}\left[5^{\frac{3}{2}} - 2^{\frac{3}{2}} \right] + \int_{\tan^{-1}(1)}^{\tan^{-1}(2)} \sec^3\theta\, d\theta$$

$$= -\frac{1}{6}\left[5^{\frac{3}{2}} - 2^{\frac{3}{2}} \right] + \left(\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| \Big|_{\tan^{-1}(1)}^{\tan^{-1}(2)} \right)$$

$u = \tan\theta, \ du = \sec^2\theta\, d\theta$

≈ 0.418.