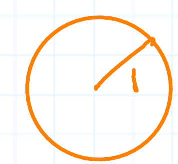
Volume of n-ball

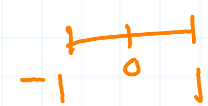
$$B_n(r) = \{ x \in \mathbb{R}^n : \sum x_i^2 \le r^2 \}$$

By change of variables, we have

$$B_n(r) = r^n B_n(1) \qquad \text{(HW)}$$

Define $v_n = \text{vol } B_n(1)$. Thus, $v_1 = 2$ and $v_2 = \pi$.



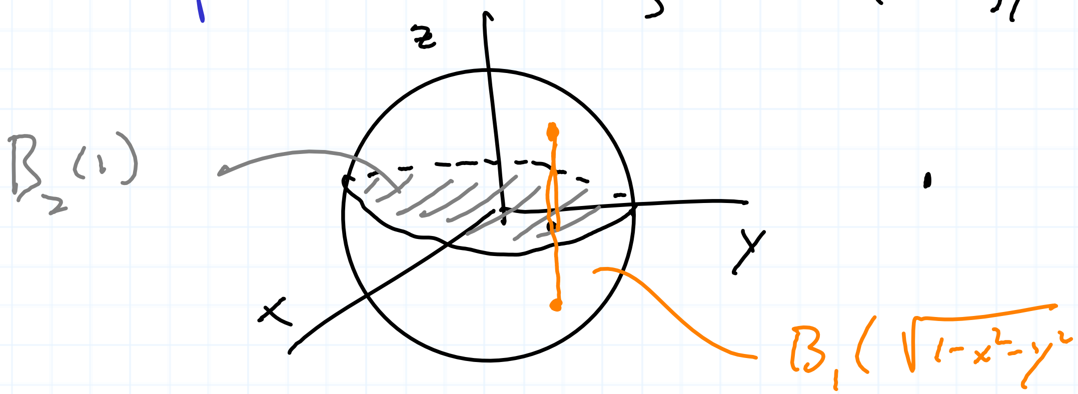


Note: $B_n(1) = \{ (x_1, \dots, x_n) : x_3^2 + x_4^2 + \dots + x_n^2 \le \overbrace{1 - x_1^2 - x_2^2}^{r^2} \}$

$$= \{ (x_1, \dots, x_n) : (x_3, \dots, x_n) \in B_{n-2}\left(\sqrt{1 - x_1^2 - x_2^2}\right), \ (x_1, x_2) \in B_2(1) \}$$

$$= \{ (x_1, x_2) \times B_{n-2}\left(\sqrt{1 - x_1^2 - x_2^2}\right) : (x_1, x_2) \in B_2(1) \}$$

Example $n=3$ $\quad B_3(1) = \{ (x,y) \times B_1(\sqrt{1-x^2-y^2}) : (x,y) \in B_2(1) \}$



Fubini: $\displaystyle v_n = \int_{B_n(1)} 1 = \int_{(x_1,x_2)\in B_2(1)} \int_{B_{n-2}(\sqrt{1-x_1^2-x_2^2})} 1$

$$= \int_{(x_1,x_2)\in B_2(1)} \mathrm{vol}(B_{n-2}(\sqrt{1-x_1^2-x_2^2})) = \int_{(x_1,x_2)\in B_2(1)} \left(\sqrt{1-x_1^2-x_2^2}\right)^{n-2} \mathrm{vol}(B_{n-2}(1))$$

$$= \int_{(x_1,x_2)\in B_2(1)} (1-x_1^2-x_2^2)^{\frac{n-2}{2}} v_{n-2}$$

Polar coordinates

$$= \int_0^{2\pi}\int_0^1 (1-r^2\cos^2-r^2\sin^2)^{\frac{n-2}{2}}\, r\, v_{n-2}\, dr\, d\theta$$

$$= \int_0^{2\pi} (1-r^2)^{\frac{n-2}{2}}\, r\, v_{n-2}\, dr\, d\theta$$

$$\frac{n-2}{2}+1 = \frac{n}{2}$$

$$= \int_0^{2\pi} \left(-\frac{1}{n}(1-r^2)^{\frac{n}{2}} \Big|_0^1 \right) v_{n-2}\, d\theta$$

$$= \int_0^{2\pi} \frac{1}{n} v_{n-2}\, d\theta = \frac{2\pi}{n} v_{n-2}.$$

Hence, $\star\; v_n = \frac{2\pi}{n} v_{n-2} \;\star$.

Starting from $v_1 = 2$: $v_1 = 2$, $v_3 = \frac{2\pi}{3} \cdot 2 = \frac{4}{3}\pi$, etc.

Starting from $v_2 = \pi$: $v_4 = \frac{2\pi}{4}\pi = \frac{\pi^2}{2}$, $v_6 = \frac{2\pi}{6}\left(\frac{\pi^2}{2}\right) = \frac{\pi^3}{6}$

In general: $v_{2n+1} = \frac{2^{2n+1}\pi^n n!}{(2n+1)!}$, $v_{2n} = \frac{\pi^n}{n!}$.

n	0	1	2	3	4	5	6	7	8
$\approx v_n$	1	2	3.14	4.18	4.93	5.26	5.18	4.71	4.05

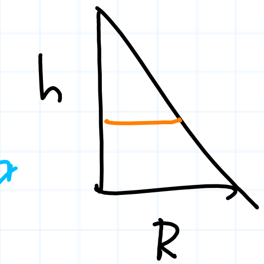
Challenge: Show $\lim_{n\to\infty} v_n = 0$.

Why should this be?

Fubini + change of coordinates practice

1. Volume of right cone  with base a circle of radius R and height h.

$0 \le z \le h, \quad 0 \le \theta \le 2\pi, \quad 0 \le r \le R - z\frac{R}{h}$

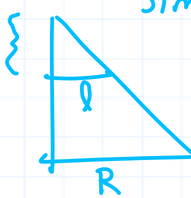


$$\mathrm{vol}(\triangle) = \int_{\triangle} 1 = \int_0^h \int_0^{2\pi} \int_0^{R - z\frac{R}{h}} r \, dr \, d\theta \, dz$$

$$= \int_0^h \int_0^{2\pi} \left(R - z\frac{R}{h}\right)^2 \Big/ 2 \, d\theta \, dz = \pi \int_0^h \left(R - z\frac{R}{h}\right)^2 dz$$

$$= -\frac{\pi h}{3R}\left(R - z\frac{R}{h}\right)^3 \Big|_0^h = 0 - \left(-\frac{\pi}{3}\frac{h}{R} R^3\right) = \pi R^2 h / 3.$$

$= \frac{1}{3}$ volume of cylinder 



Similar triangles:

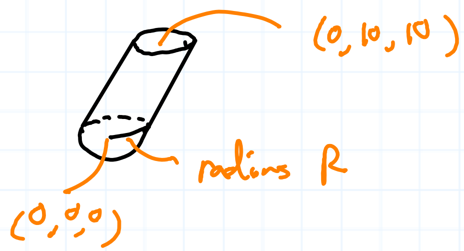
$$\frac{l}{h-z} = \frac{R}{h} \Rightarrow l = \frac{R}{h}(h-z) = R - z\frac{R}{h}$$

2. Area inside an ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ [Hint: change coordinates ○ → ⬭ → ℝ

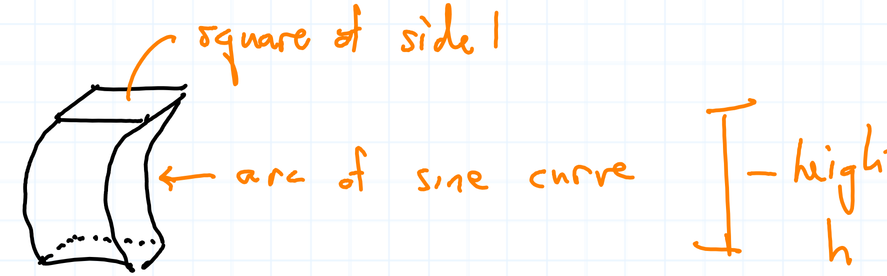
3. Volume of an ellipsoid $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$.

square of side 1

4. Volume of: (a)



(b)



5. $\int_1^2 \int_0^{\ln x} (x-1)\sqrt{1+e^{2y}}\, dy\, dx.$

6. Volume enclosed by $z^2 = x^2+y^2$ and the plane $x+y+4z=4$.