Math 212 Quiz: Two questions from our webpage.

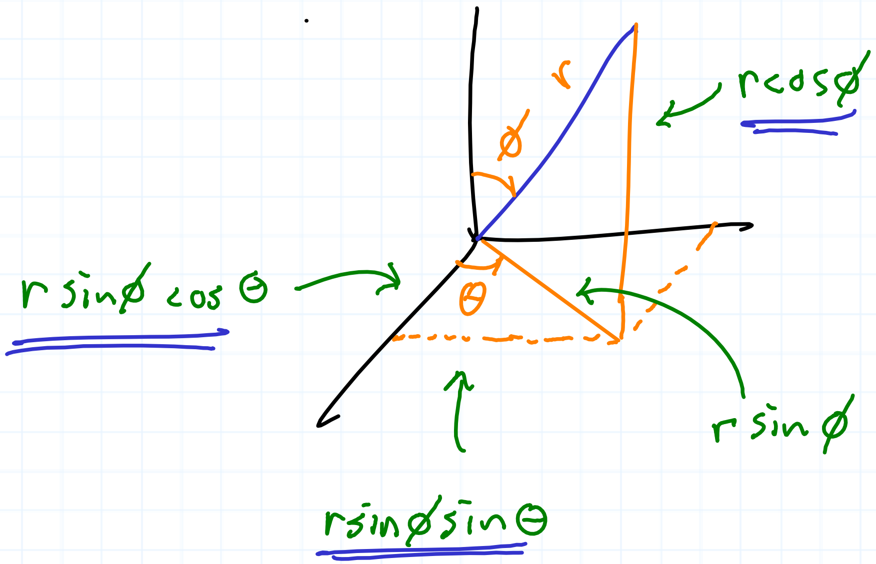
Common changes of coordinates

* polar coordinates $\Phi(r,t) = (r\cos t, r\sin t)$

  Look for circles or functions involving x^2+y^2.

* spherical coordinates $S(r,\theta,\phi) = (r\cos\theta\sin\phi,\ r\sin\theta\sin\phi,\ r\cos\phi)$

  $r \geq 0,\quad 0 \leq \theta \leq 2\pi,\quad 0 \leq \phi \leq \pi.$



Stretching factor:

$$|\det J_S| = \left|\det \begin{array}{ccc} \begin{bmatrix} \cos\theta\sin\phi & -r\sin\theta\sin\phi & r\cos\theta\cos\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\sin\phi \\ r\cos\phi & 0 & -r\sin\phi \end{bmatrix} \\ \uparrow \qquad\qquad \uparrow \qquad\qquad \uparrow \\ \frac{\partial S}{\partial r} \qquad\quad \frac{\partial S}{\partial \theta} \qquad\quad \frac{\partial S}{\partial \phi} \end{array}\right|$$

$$= |r^2\sin\phi| = r^2\sin\phi$$

Examples Integrate $f(x,y,z) = \sqrt{x^2+y^2+z^2}$ over the solid sphere of radius R.

1. $$\int_{\oplus} f = \int_{\square} f\circ S \; r^2 \sin\phi = \int_0^{\pi}\int_0^{2\pi}\int_0^{R} \sqrt{r^2}\; r^2 \sin\phi \; dr\, d\theta\, d\phi$$

$$= \int_0^{\pi}\int_0^{2\pi}\int_0^{R} r^3 \sin\phi \; dr\, d\theta\, d\phi = \int_0^{\pi}\int_0^{2\pi} \frac{R^4}{4} \sin\phi \; d\theta\, d\phi = 2\pi \frac{R^4}{4} \int_0^{\pi} \sin\phi \, d\phi$$

$$= \pi R^4/\cancel{2} \; \left[-\cos\phi \,\Big|_0^{\pi}\right] = \pi R^4/\cancel{2} \; (\cancel{2}) = \pi R^4.$$

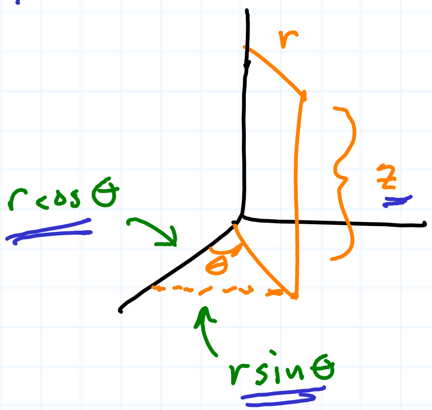
2. The volume of a sphere of radius R

$$\int_{\oplus} 1 = \int_{\square} r^2 \sin\phi = \int_0^{\pi}\int_0^{2\pi}\int_0^{R} r^2 \sin\phi \; dr\, d\theta\, d\phi$$

$$= \int_0^{\pi}\int_0^{2\pi} \frac{R^3}{3} \sin\phi \; d\theta\, d\phi = 2\pi \frac{R^3}{3} \int_0^{\pi} \sin\phi \, d\phi = 2\pi \frac{R^3}{3} \left(-\cos\phi \,\Big|_0^{\pi}\right)$$

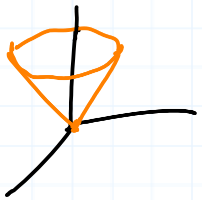
$$= 4\pi R^3/3.$$

* Cylindrical coordinates $c(r,\theta,z) = (r\cos\theta, r\sin\theta, z)$ $\quad 0 \le r,\ 0 \le \theta \le 2\pi$



$$|J_c| = r$$

1. Integrate $\int_S \sqrt{x^2+y^2}$ where S is the region bounded by $1 \ge z^2 \ge x^2+y^2$,

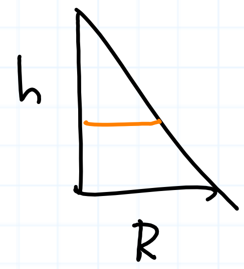


$$\int_{\bigtriangledown} \sqrt{x^2+y^2} = \int_0^1 \int_0^z \int_0^{2\pi} \sqrt{r^2}\; r \, d\theta\, dr\, dz = 2\pi \int_0^1 \int_0^z r^2 \, dr\, dz$$

$$= 2\pi \int_0^1 \left(\frac{r^3}{3} \Big|_0^z \right) dz = 2\pi \int_0^1 \frac{z^3}{3}\, dz = 2\pi \frac{z^4}{4\cdot 3} \Big|_0^1 = \frac{\pi}{6}.$$

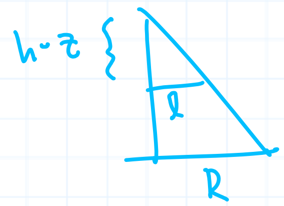
2. Volume of right cone  with base a circle of radius R and height h.

$$0 \le z \le h, \quad 0 \le \theta \le 2\pi, \quad 0 \le r \le R - z\frac{R}{h}$$



$$\text{vol}(\triangle) = \int_{\triangle} 1 = \int_0^h \int_0^{2\pi} \int_0^{R - z\frac{R}{h}} r \, dr \, d\theta \, dz$$

Similar triangles:



$$\frac{l}{h-z} = \frac{R}{h} \Rightarrow l = R - z\frac{R}{h}.$$

$$= \int_0^h \int_0^{2\pi} \left(R - z\frac{R}{h}\right)^2 \Big/ 2 \, d\theta \, dz = \pi \int_0^h \left(R - z\frac{R}{h}\right)^2 dz$$

$$= -\frac{\pi h}{3R}\left(R - z\frac{R}{h}\right)^3 \Big|_0^h = 0 - \left(-\frac{\pi}{3}\frac{h}{R} R^3\right) = \pi R^2 h / 3. \quad = \frac{1}{3} \text{ volume of cylinder}$$