Math 212

Determinants

$$M = \begin{array}{c c} & \begin{array}{ccc} + & - & + \end{array} \\ \begin{array}{c} + \\ - \\ + \end{array} & \begin{bmatrix} 1 & 0 & 3 \\ -1 & 2 & 2 \\ 4 & 1 & 4 \end{bmatrix} \end{array}$$

Expand along 2^{nd} column

$$\det M = -0 \det \begin{bmatrix} 1 & 0 & 3 \\ -1 & 2 & 2 \\ 4 & 1 & 4 \end{bmatrix} + 2 \det \begin{bmatrix} 1 & 0 & 3 \\ -1 & 2 & 2 \\ 4 & 1 & 4 \end{bmatrix} - 1 \cdot \det \begin{bmatrix} 1 & 0 & 3 \\ -1 & 2 & 2 \\ 4 & 1 & 4 \end{bmatrix}$$

$$= 0 + 2 \det \begin{bmatrix} 1 & 3 \\ 4 & 4 \end{bmatrix} - \det \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} = 2(1 \cdot 4 - 3 \cdot 4) - (1 \cdot 2 - 3(-1))$$

$$= 2(-8) - 5 = -21.$$

Expand along 3^{rd} row

$$\det M = 4 \det \begin{bmatrix} 1 & 0 & 3 \\ -1 & 2 & 2 \\ 4 & 1 & 4 \end{bmatrix} - 1 \cdot \det \begin{bmatrix} 1 & 0 & 3 \\ -1 & 2 & 2 \\ 4 & 1 & 4 \end{bmatrix} + 4 \det \begin{bmatrix} 1 & 0 & 3 \\ -1 & 2 & 2 \\ 4 & 1 & 4 \end{bmatrix}$$

$$= 4 \begin{vmatrix} 0 & 3 \\ 2 & 2 \end{vmatrix} - \begin{vmatrix} 1 & 3 \\ -1 & 2 \end{vmatrix} + 4 \begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} = 4(0 \cdot 2 - 3 \cdot 2) - (1 \cdot 2 - 3(-1)) + 4(1 \cdot 2 - 0(-1)) = -21$$

$$= 2(-8) - 5 = 21.$$

Think of the determinant as a function on the rows of a matrix:

$$\det \begin{bmatrix} 1 & 3 \\ 2 & -4 \end{bmatrix} = \det\big(\underbrace{(1,3)}_{r_1}, \underbrace{(2,-4)}_{r_2}\big) = \det(r_1, r_2).$$

Then det is 1 multilinear:

$$\det(r_1 + ar, r_2) = \det(r_1, r_2) + a\det(r, r_2)$$
$$\det(r_1, ar + r_2) = a\det(r_1, r) + \det(r_1, r_2)$$

2 alternating: $\det(r, r) = 0$, whence $\det(r_1, r_2) = -\det(r_2, r_1)$.

[$0 = \det(r_1 + r_2, r_1 + r_2) \overset{\text{by multilinearity}}{=} \det(r_1, r_1) + \det(r_1, r_2) + \det(r_2, r_1) + \det(r_2, r_2)$

$= 0 + \det(r_1, r_2) + \det(r_2, r_1) + 0 \Rightarrow \det(r_1, r_2) = -\det(r_2, r_1)$]

and det of the identity matrix is 1: $\det((1,0),(0,1)) = 1$.

Consequence: Adding a multiple of one row to another does not change the value of the determinant:

$$\det(r_1 + a r_2, r_2) = \det(r_1, r_2) + a\,\overbrace{\det(r_2, r_2)}^{0} = \det(r_1, r_2)$$

See the handout on determinants at our website.

Determinants and permutations (rook placements)

$$
\begin{array}{c}\begin{bmatrix} 1 & 0 & 3 \\ -1 & 2 & 2 \\ 4 & 1 & 4 \end{bmatrix} \\ 1\ 2\ 3 \\ (-1)^0 = + \end{array}
\quad
\begin{array}{c}\begin{bmatrix} 1 & 0 & 3 \\ -1 & 2 & 2 \\ 4 & 1 & 4 \end{bmatrix} \\ 2\ 1\ 3 \\ (-1)^1 = - \end{array}
\quad
\begin{array}{c}\begin{bmatrix} 1 & 0 & 3 \\ -1 & 2 & 2 \\ 4 & 1 & 4 \end{bmatrix} \\ 3\ 2\ 1 \\ (-1)^1 = - \end{array}
\quad
\begin{array}{c}\begin{bmatrix} 1 & 0 & 3 \\ -1 & 2 & 2 \\ 4 & 1 & 4 \end{bmatrix} \\ 1\ 3\ 2 \\ (-1)^1 = - \end{array}
\quad
\begin{array}{c}\begin{bmatrix} 1 & 0 & 3 \\ -1 & 2 & 2 \\ 4 & 1 & 4 \end{bmatrix} \\ 3\ 1\ 2 \\ (-1)^2 = + \end{array}
\quad
\begin{array}{c}\begin{bmatrix} 1 & 0 & 3 \\ -1 & 2 & 2 \\ 4 & 1 & 4 \end{bmatrix} \\ 2\ 3\ 1 \\ (-1)^2 = + \end{array}
$$

flips: $(-1)^0 = +$, $(-1)^1 = -$, $(-1)^1 = -$, $(-1)^1 = -$, $(-1)^2 = +$, $(-1)^2 = +$

$$\text{det: } +1\cdot 2\cdot 4 - (-1)\cdot 0\cdot 4 - 4\cdot 2\cdot 3 - 1\cdot 1\cdot 2 + 4\cdot 0\cdot 2 + (-1)\cdot 1\cdot 3$$
$$= 8 - 0 - 24 - 2 + 0 - 3 = -21.$$