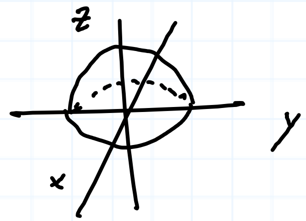
Fubini example

Find the volume of $S = \{ (x,y,z) \in \mathbb{R}^3 : 0 \le z \le 1 - x^2 - y^2 \}$

(a piece of a solid paraboloid).



Solution/ Let $B = \underbrace{[-1,1] \times [-1,1]}_{J} \times [0,1]$.

Then $\mathrm{vol}(S) = \int_S \chi_S \underset{\text{Fubini}}{=} \int_{z=0}^{1} \left(\int_J \chi_S \right) = \int_{z=0}^{1} (\text{area of circle of radius } \sqrt{1-z})$

$$= \int_0^1 \pi (1-z)\, dz = \pi z - \tfrac{1}{2}\pi z^2 \Big|_0^1 = \frac{\pi}{2}. \quad \Box$$

Change of variables formula

there's a cts. path inside K between any two pts. of K

boundary

Thm. Let $K \subseteq \mathbb{R}^n$ be a compact and connected set with $\mathrm{vol}(\partial K) = 0$.

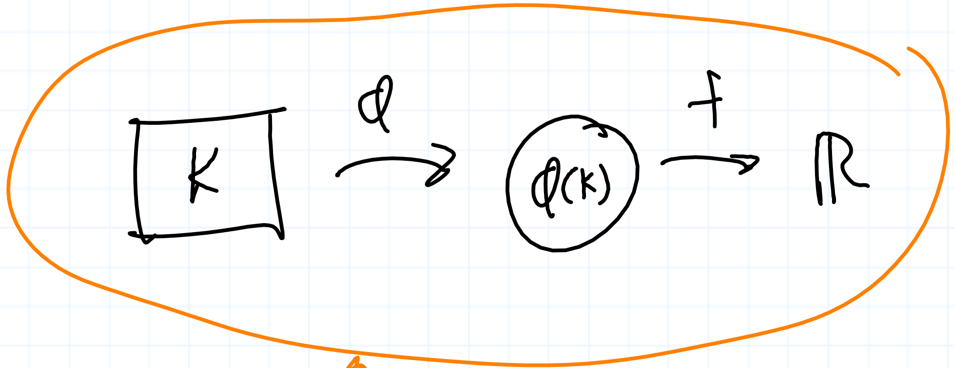
Let $U \subseteq \mathbb{R}^n$ be an open set such that $U \supseteq K$. Let

$\varphi : U \to \mathbb{R}^n$ be a differentiable function with continuous first partials.
s.t. φ is injective on the interior of $K^\circ := K \setminus \partial K$, and
$\det J\varphi \neq 0$ on K°. Suppose

$$f : \varphi(K) \to \mathbb{R}$$

is continuous. Then

$$\int_{\varphi(K)} f = \int_K f \circ \varphi \, |\det J\varphi| .$$



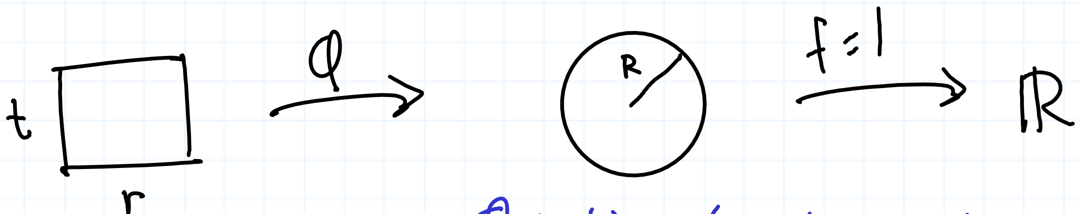
Remember this picture

Examples

1. Find the area of a circle of radius R.

Solution / Parametrize the circle!

$0 \leq t \leq 2\pi$



$0 \leq r \leq R$

$\varphi(r,t) = (r\cos t, r\sin t)$

By the change of variables formula:

$$\mathrm{vol}(\bigcirc_R) = \int_{\bigcirc_R} 1 = \int_{\square} f\circ\phi \, |\det J(\phi)|$$

$f\circ\phi = 1\circ\phi = 1$

$$J\phi = \begin{bmatrix} \cos t & -r\sin t \\ \sin t & r\cos t \end{bmatrix}$$

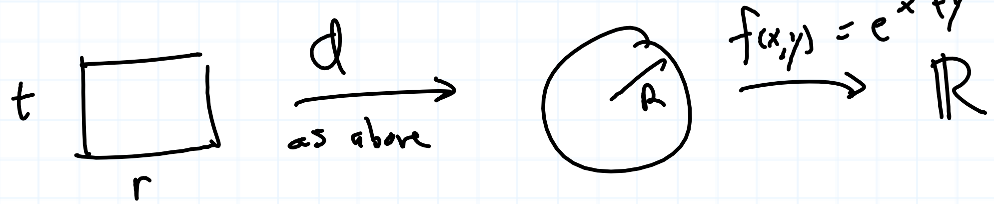
$\det J\phi = r\cos^2 t + r\sin^2 t = r$

$$= \int_{\square} r = \int_0^R \int_0^{2\pi} r \, dt \, dr$$

$$= \int_0^R 2\pi r \, dr = \pi r^2 \Big|_0^R$$

$$= \pi R^2.$$

2. Integrate $f(x,y) = e^{x^2+y^2}$ over a circle of radius R.



$$\int_{\bigcirc_R} e^{x^2+y^2} = \int_{\square} f\circ\phi \, |J\phi| = \int_{\square} f(r\cos t, r\sin t)\cdot r$$

$$= \int_{\square} e^{r^2\cos^2 t + r^2\sin^2 t}\, r = \int_{\square} re^{r^2}$$

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$$= \int_0^R \int_0^{2\pi} re^{r^2}\, dt\, dr = 2\pi \int_0^R re^{r^2}\, dr = \pi e^{r^2} \Big|_0^R$$

$$= \pi e^{R^2} - \pi .$$