Math 212

Quiz Let $B \subseteq \mathbb{R}^n$ be a box, and let $f: B \to \mathbb{R}$ be continuous. Show that f is integrable.

Statement of Fubini's theorem (2-d version)

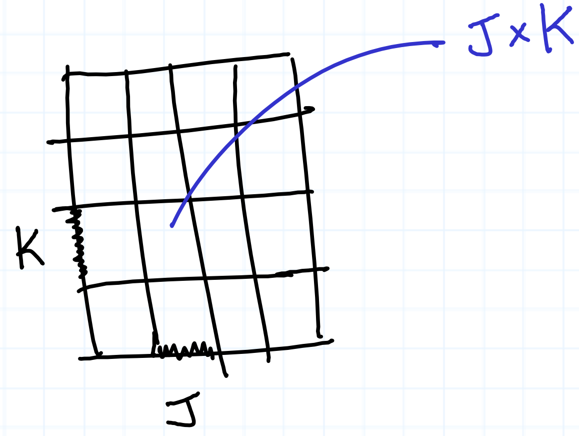
Let $f: [a,b] \times [c,d] \to \mathbb{R}$ be integrable. For each $x \in [a,b]$, suppose $f(x,y)$ is integrable on $[c,d]$ as a function of y. Then

$$\int_{[a,b]\times[c,d]} f = \int_a^b \int_c^d f(x,y)\, dy\, dx$$

(where the right-hand side is an iterated integral).

Pf/ Let P and Q be partitions of $[a,b]$ and $[c,d]$, respectively. So $P \times Q$ is an arbitrary partition of $B := [a,b] \times [c,d]$. For each $x \in [a,b]$, define $g(x) = \int_c^d f(x,y)\,dy$. Then

$$L(g,P) = \sum_J m_J(g)\,\mathrm{vol}(J).$$



For $x \in J$, we have

$$g(x) = \int_c^d f(x,y)\,dy = \sum_K \int_K f$$

$$\geq \sum_K \int_K m_{J\times K}(f) = \sum_K m_{J\times K}(f)\,\mathrm{vol}(K).$$

So for $x \in J$, we see $\sum_K m_{J\times K}(f)\,\mathrm{vol}(K)$ is a lower bound for $g(x)$, hence, less than or equal to the greatest lower bound, $m_J(g)$. Therefore, 3

$$L(g,P) = \sum_J m_J(g)\,\mathrm{vol}(J) \geq \sum_J \sum_K m_{J\times K}(f)\,\underbrace{\mathrm{vol}(K)\,\mathrm{vol}(J)}_{\mathrm{vol}(J\times K)}$$

$$= \sum_{J\times K} m_{J\times K}(f)\,\mathrm{vol}(J\times K) = L(f, P\times Q).$$

Similarly, $U(g,P) \leq U(f, P\times Q)$. So

$$L(f,P\times Q) \leq L(g,P) \leq U(g,P) \leq U(f,P\times Q) \quad \star$$

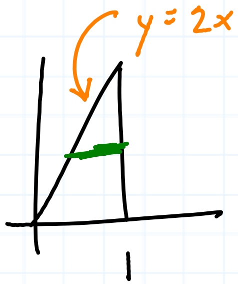
for all P, Q. Since f is integrable, given $\varepsilon > 0$, we can choose $P\times Q$ fine enough so that $U(f,P\times Q) - L(f,P\times Q) < \varepsilon$. Therefore, by $\star$, $U(g,P) - L(g,P) < \varepsilon$. This shows $\int g$ exists. But then $\int_a^b g$ and $\int_B f$ are both between $U(f, P\times Q)$ and $L(f,P\times Q)$ for all $P\times Q$. This forces

$$\int_a^b \underbrace{\left(\int_c^d f(x,y)\,dy\right)}_{g(x)} = \int_a^b g = \int_B f. \qquad \square$$

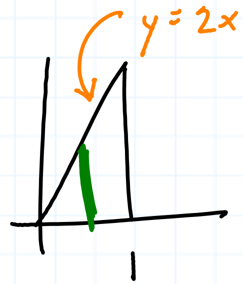
Fubini example

* Integrate $\int_0^2 \int_{y/2}^1 e^{-x^2}\,dx\,dy$

Solution/ Switch the order of integration:



$$\int_0^2 \int_{y/2}^1 e^{-x^2}\,dx\,dy = \int_0^1 \int_0^{2x} e^{-x^2}\,dy\,dx$$



$$= \int_0^1 2x\, e^{-x^2}\,dx = -e^{-x^2}\Big|_0^1 = -e^{-1} - (-e^0) = 1 - \frac{1}{e}.$$