Math 212

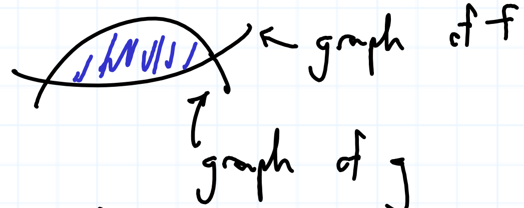
Last time **Thm.** If $f: B \to \mathbb{R}$ is cts. except on a set of volume 0, then f is integrable.

Def. Let $S \subseteq \mathbb{R}^n$. The **closure** of S, denoted $\overline{S}$, is the intersection of all closed sets containing S. (It's the closed set formed by adding all the limit points of S to S.) The **boundary** of S, denoted ∂S, is

$$\partial S = \overline{S} \cap \overline{(\mathbb{R}^n \setminus S)}.$$

Cor. If $S \subseteq \mathbb{R}^n$ is bounded, $\mathrm{vol}(\partial S) = 0$, and $f: S \to \mathbb{R}$ is cts. and bounded, then $\int_S f$ exists. In particular, $\mathrm{vol}(S)$ exists.

Remark. Our text shows (6.5.3) that the graph of a continuous function has volume 0. Thus, regions such as:



have volume.

$M = \{(x,y) : f(x) \leq y \leq g(x)\}$

Thm. If $f, g : B \to \mathbb{R}$ are integrable and $f = g$ except on a set of volume 0, then $\int f = \int g$.

Pf/ We know $f-g$ is integrable and $\int (f-g) = \int f - \int g$.
Let $h := f-g$. Then $h = 0$ except on a set of volume 0.
So the theorem from last time says h is integrable. Looking back at the proof of that theorem shows $\int h = 0$. $\square$

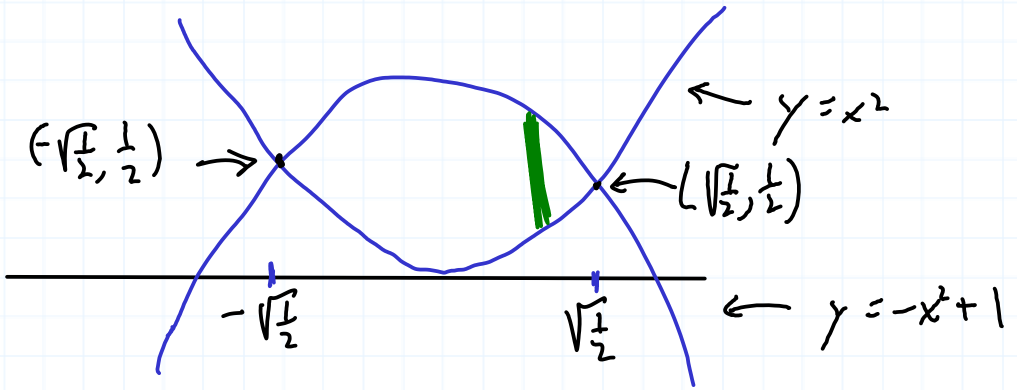
Fubini's thm. (by example)

* $$\int_{[0,1]\times[0,2]} xy^2 + y = \int_0^1 \int_0^2 xy^2 + y \, dy\, dx = \int_0^1 \left[\tfrac{1}{3}xy^3 + \tfrac{1}{2}y^2 \Big|_0^2 \right] dx$$
$$= \int_0^1 \tfrac{8}{3}x + 2 \, dx = \tfrac{4}{3}x^2 + 2x \Big|_0^1 = \tfrac{4}{3} + 2 = \tfrac{10}{3}$$

* $$\int_{[0,1]\times[0,2]} xy^2 + y = \int_0^2 \int_0^1 xy^2 + y \, dx\, dy = \int_0^2 \left[\tfrac{1}{2}x^2y^2 + xy \Big|_0^1 \right] dy$$
$$= \int_0^2 \tfrac{1}{2}y^2 + y \, dy = \tfrac{1}{6}y^3 + \tfrac{1}{2}y^2 \Big|_0^2 = \tfrac{4}{3} + 2 = \tfrac{10}{3}.$$

• Find the area between $y=x^2$ and $y=-x^2+1$ for $x \in \left[-\sqrt{\tfrac{1}{2}}, \sqrt{\tfrac{1}{2}}\right]$.

method 1.



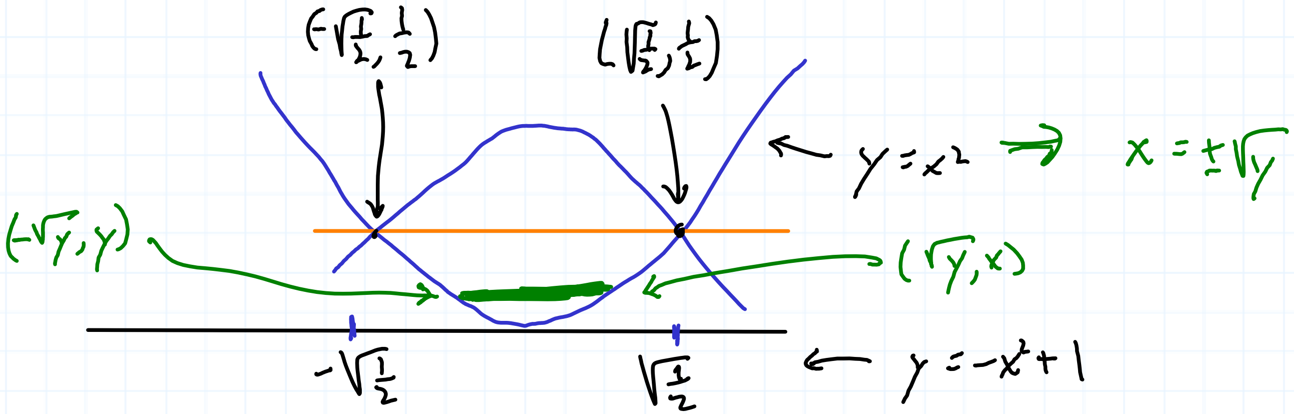
Let S be the enclosed region. Then $\mathrm{vol}(S) = \int_S \chi_S$. By Fubini,

$$\mathrm{vol}(S) = \int_{-\sqrt{\frac{1}{2}}}^{\sqrt{\frac{1}{2}}} \int_{x^2}^{-x^2+1} 1 \, dy \, dx = \int_{-\sqrt{\frac{1}{2}}}^{\sqrt{\frac{1}{2}}} \left(y \Big|_{x^2}^{-x^2+1} \right) dx$$

$$= \int_{-\sqrt{\frac{1}{2}}}^{\sqrt{\frac{1}{2}}} \left((-x^2+1) - x^2 \right) dx = \int_{-\sqrt{\frac{1}{2}}}^{\sqrt{\frac{1}{2}}} -2x^2+1 \, dx = -\frac{2}{3}x^3 + x \Big|_{-\sqrt{\frac{1}{2}}}^{\sqrt{\frac{1}{2}}}$$

$$= \left(-\frac{1}{3}\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{2}}\right) - \left(\frac{1}{3}\sqrt{\frac{1}{2}} - \sqrt{\frac{1}{2}}\right) = \frac{4}{3}\sqrt{2}.$$

method 2



$$\mathrm{vol}(S) = 2\int_0^{\frac{1}{2}}\int_{-\sqrt{y}}^{\sqrt{y}} 1 \, dx\, dy = 2\int_0^{\frac{1}{2}} \left(x \Big|_{-\sqrt{y}}^{\sqrt{y}} \right) dy$$

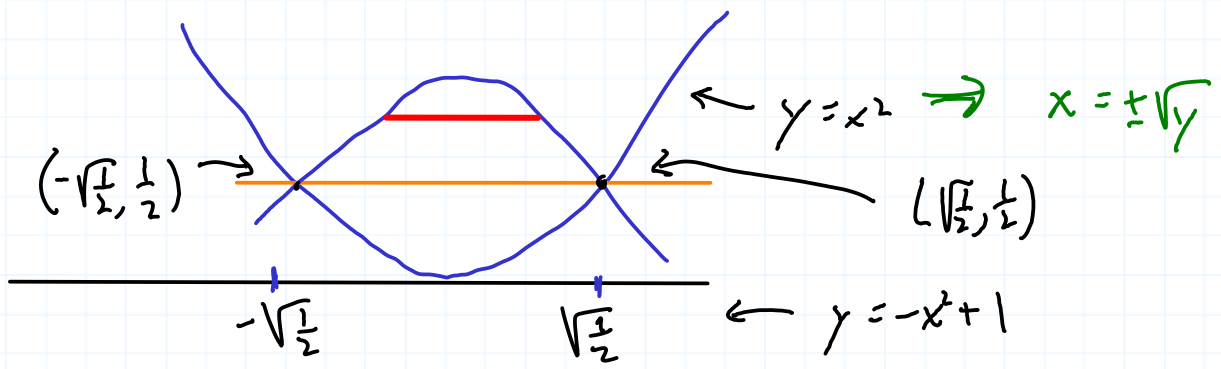
We are only integration halfway: $y \in [0, \frac{1}{2}]$, then doubling, using symmetry.

$$= 2\int_0^{\frac{1}{2}} 2\sqrt{y}\, dy = 4 \cdot \frac{2}{3} y^{\frac{3}{2}} \Big|_0^{\frac{1}{2}}$$

$$= \frac{8}{3}\left(\frac{1}{2}\right)^{\frac{3}{2}} = \frac{4}{3}\sqrt{2}. \quad \checkmark$$

method 3

(horizontal slices w/out appealing to symmetry)



$$\mathrm{vol}(S) = \underbrace{\int_0^{\frac12}\int_{-\sqrt{y}}^{\sqrt{y}} 1 \, dx\, dy}_{\text{as befor}} + \underbrace{\int_{\frac12}^{1}\int_{-\sqrt{1-y}}^{\sqrt{1-y}} 1 \, dx\, dy}_{\text{area of upper half}} = \text{etc.} = \frac{4}{3}\sqrt{2}.$$

Note: y goes from $\frac12$ to 1

and $y = -x^2 + 1 \Rightarrow x = \pm\sqrt{1-y}$.