Math 212

Last time $S \subseteq \mathbb{R}^n$ bounded, $f : S \to \mathbb{R}$ bounded. Pick any box $B \supseteq S$, and extend f by 0 to get a function B: $\tilde{f} : B \to \mathbb{R}$

$$x \mapsto \begin{cases} f(x) & x \in S \\ 0 & x \in B \setminus S. \end{cases}$$

Say f is *integrable* if $\tilde{f}$ is integrable, in which case $\int_S f := \int_B \tilde{f}$.

Def. Let $S \subseteq \mathbb{R}^n$. The *characteristic function* or *indicator function* for S is $\chi_S : \mathbb{R}^n \longrightarrow \mathbb{R}$

$$x \mapsto \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S. \end{cases}$$

If S is bounded, define the *volume* of S by

$$\operatorname{vol}(S) := \int_S \chi_S .$$

If the integral does not exist, then S does not have volume.

This does not mean that S has volume 0.

Example $S = \mathbb{Q} \cap [0,1] = \{ x \in [0,1] : x \text{ is rational} \}$ does not have volume. $\left(\underline{\int} \chi_S = 0 \neq 1 = \overline{\int} \chi_S \right)$

Thm. Let B be a box in $\mathbb{R}^n$ and let $f : B \to \mathbb{R}$ be a bounded function. Then f is integrable if it is continuous except on a set of volume 0.

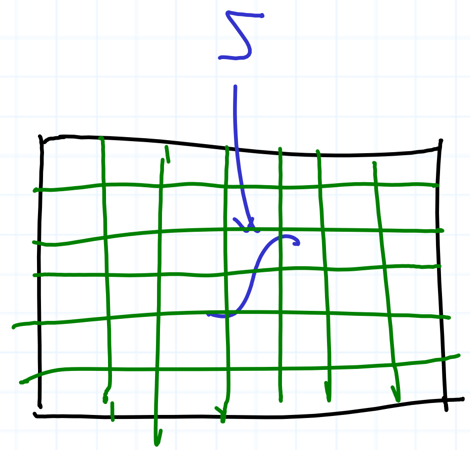
Lemma (volume zero criterion) Let $S \subseteq \mathbb{R}^n$ be a bounded set contained in a box B. Then $\text{vol}(S) = 0$ iff $\forall \varepsilon > 0$, $\exists$ partition P of B s.t.

$$\sum_{J : J \cap S \neq \emptyset} \text{vol}(J) < \varepsilon.$$

Pf/ Exercise. $\square$

Pf. of thm./ Let $\varepsilon > 0$.

Let S be the set of volume zero



where f is not continuous. Pick a partition P so that

$$\sum_{J : J \cap S \neq \emptyset} \operatorname{vol}(J) < \frac{\varepsilon}{4b}$$

where $b > 0$ is bound on f. Let

$$\text{☹} = \bigcup_{J : J \cap S \neq \emptyset} J \quad \text{and} \quad \text{☺} = \bigcup_{J : J \cap S = \emptyset} J .$$

Since ☺ is compact and f is cts. on ☺, it follows that f is uniformly cts. on ☺. So $\exists$ refinement P' of P s.t. if $J' \in \operatorname{Boxes}(P')$ and $J' \subseteq \text{☺}$, then $x, y \in J' \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{2\operatorname{vol}(B)}$.

One further point to note: since $|f(x)| < b \ \forall\, x \in B$, we have

$|M_J(f)| \leq b$ and $|m_J(f)| \leq b$, hence, $0 \leq M_J(f) - m_J(f) \leq 2b$.

Finally,

$$U(f,P') - L(f,P') = \sum_{J'} \left[M_{J'}(f) - m_{J'}(f)\right] \mathrm{vol}\,(J')$$

$$= \sum_{J' : J' \subseteq \smiley} \left[M_{J'}(f) - m_{J'}(f)\right] \mathrm{vol}\,(J') + \sum_{J' : J' \subseteq \frownie} \left[M_{J'}(f) - m_{J'}(f)\right] \mathrm{vol}\,(J')$$

$$\leq \frac{\varepsilon}{2\,\mathrm{vol}(B)} \sum_{\smiley} \mathrm{vol}\,(J') + 2b \sum_{\frownie} \mathrm{vol}\,(J')$$

$$< \frac{\varepsilon}{2\,\mathrm{vol}(B)} \cdot \mathrm{vol}\,(B) + 2b \cdot \frac{\varepsilon}{4b} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad \square$$

Remarks

- A similar argument shows that if $f, g : B \to \mathbb{R}$ are bounded functions, f is integrable, and $f = g$ except on a set of volume 0, then g is integrable and $\int f = \int g$.

- If $S \subseteq \mathbb{R}^n$ is bounded, $\mathrm{vol}(\partial S) = 0$, and $f: S \to \mathbb{R}$ is cts. and bounded, then $\int_S f$ exists. In particular, letting $f = \chi_S$, we see that $\mathrm{vol}(S)$ exists.

∂S = boundary of S = $\overline{S} \cap \overline{(S^c)}$

closure of S

closure of the complement of S

Pf/ This follows directly from the theorem we just proved.