Today: • more properties of integrals
• def. of integral over any bounded set.

Math 212

☆ Announce Thursday talk

Quiz. Let $B = I_1 \times \cdots \times I_n$ be a box in $\mathbb{R}^n$, and suppose $f : B \to \mathbb{R}$ is a bounded function. What does it mean to say f is integrable. (You may assume that I know what a partition of B is.)

Question. What is the main idea behind the fact that $L(f,P) \leq U(f,Q)$ $\forall$ partitions P, Q?

Thm. (linearity of $\int$) Let $f, g : B \to \mathbb{R}$ be integrable functions, and let $c \in \mathbb{R}$. Then $cf + g$ is integrable and

$$\int_B (cf+g) = c\int_B f + \int_B g.$$

Sketch of proof / Step 1: cf is integrable if $c \geq 0$ and $\int cf = c\int f$.

Step 2: $-f$ is integrable and $\int(-f) = -\int f$.

Step 3: $f+g$ is integrable and $\int(f+g) = \int f + \int g$.

Proof of step 3: Given $\varepsilon > 0$, choose a partition P of B so that

$$U(f,P) - L(f,P) < \frac{\varepsilon}{2} \quad \text{and} \quad U(g,P) - L(g,P) < \frac{\varepsilon}{2}$$

For each subbox J of P, we have $m_J(f+g) = \inf\{f(x)+g(x) : x \in J\}$

$$\geq \inf\{f(x) : x \in J\} + \inf\{g(x) : x \in J\} = m_J(f) + m_J(g).$$

similar to this weeks HW: $\inf(A+B) = \inf A + \inf B$

Similarly, $M_J(f+g) \leq M_J(f) + M_J(g)$.

Hence, $L(f+g,P) \geq L(f,P) + L(g,P)$ and $U(f+g,P) \leq U(f,P) + U(g,P)$.

Therefore, $U(f+g,P) - L(f+g,P) \leq [U(f,P) + U(g,P)] - [L(f,P) + L(g,P)]$

$$= [U(f,P) - L(f,P)] + [U(g,P) - L(g,P)] < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This shows $f+g$ is integrable. Since $L(f,P) \le \int f \le U(f,P)$ and $L(g,P) \le \int g \le U(g,P)$, we have $L(f+g,P) \le \int f + \int g \le U(f+g,P)$.

Since $L(f+g,P) \le \int(f+g) \le U(f+g,P)$, too, we see

$$\left| \int(f+g) - \left[\int f + \int g\right] \right| \le U(f+g,P) - L(f+g,P) < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\int(f+g) = \int f + \int g$. $\square$

Thm. If $f,g : B \to \mathbb{R}$ are integrable and $f(x) \le g(x)$ $\forall x \in B$ then $\int f \le \int g$.

Pf/ HW for next week. $\square$

Integration over bounded sets

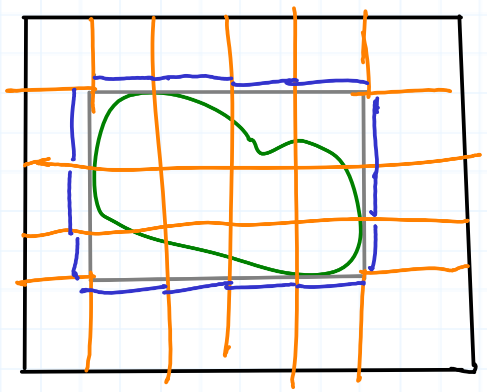
Def. Let $S \subseteq \mathbb{R}^n$ be any bounded set, and let $f: S \to \mathbb{R}$ be a bounded function. To define the integral of f over S, choose any box $B \supseteq S$ and extend f by zero to B:

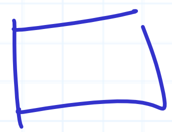
$$\tilde{f}: B \longrightarrow \mathbb{R}$$
$$x \longmapsto \begin{cases} f(x) & \text{if } x \in S \\ 0 & \text{if } x \in B \setminus S. \end{cases}$$

Then f is *integrable* on S if $\tilde{f}$ is integrable on B, and in that case $\int_S f := \int_B \tilde{f}$.

Remark: The choice of B in the definition does not matter. Let B^* be the intersection of all boxes containing B^*. Let f^* be the extension of f by zero to B^*. For any other box $\tilde{B} \supseteq S$, Let $\tilde f$ be the extension of f by zero to $\tilde B$. It suffices to show $\tilde f$ is integrable on $\tilde B$ iff f^* is integrable on B^*.

To see this, note that given partitions $\tilde P$, P^* of $\tilde B$ and B^*, respectively,





$= B^* \subseteq \tilde B$

by refining partitions, we may assume $\tilde P \supseteq P^*$. Then $L(\tilde f, \tilde P) - L(f^*, P^*) = \sum_J' m_J(f)\,\mathrm{vol}(J)$ and $U(\tilde f, \tilde P) - U(f^*, P^*) = \sum_J' M_J(f)\,\mathrm{vol}(J)$, where $\sum_J'$ means summing only over subboxes of $\tilde P$ that are not subboxes of P^* but intersect S.

By further refinement we can make $\sum_J' M_J(f)\,\mathrm{vol}(J) < \varepsilon$ for any given $\varepsilon > 0$. $\square$

Note: These intersections would be along the boundary of the subboxes