Math 212

Today: cts $\Rightarrow$ integrable

$\star$ Tomorrow's quiz:

* What does it mean to say $f$ is integrable?
* Outline the proof that $L(f,P) \le U(f,Q)$ $\forall$ partitions $P, Q$.

closed and bounded

**Thm.** Let $K \subseteq \mathbb{R}^n$ be compact, and let $f: K \to \mathbb{R}$ be continuous. Then $f$ is uniformly continuous.

Pf/ Suppose not. So $\exists \varepsilon > 0$ s.t. $\forall \delta > 0$, $\exists x, y \in K$ s.t. $|f(x) - f(y)| \geq \varepsilon$ even though $|x-y| < \delta$. Fix such an $\varepsilon$. Let $\delta_i = \frac{1}{i}$ and take $x_n$ and $y_n$ s.t. $|x_i - y_i| < \frac{1}{i}$ and $|f(x_i) - f(y_i)| \geq \varepsilon$. Since $\{x_i\}$ is a sequence in a compact set, it has a convergent subsequence $\{x_{i_j}\}$ by the Bolzano-Weierstraß property (see Jerry's notes, Thm. 2.4.11). By throwing out the other elements of $\{x_n\}$ and re-indexing, we may assume $x_i \to p$ (and $p \in K$ since $K$ is closed). Then $|x_i - y_i| < \frac{1}{i}$ $\forall i$

$\Rightarrow$ $y_i \to p$, too. By continuity, $f(x_i) \to f(p)$ and $f(y_i) \to f(p)$, contradicting the fact that $|f(x_i) - f(y_i)| \geq \varepsilon > 0$ $\forall i$. $\square$

**Thm.** $f: B \to \mathbb{R}$ cts. $\Rightarrow$ $f$ integrable.

Pf/ Since $B$ is compact, $f$ is uniformly continuous. Given $\varepsilon > 0$, take $\delta > 0$ such that $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{\text{vol}(B)}$. Let $P$ be a partition of $B$ for which each subbox has side lengths less than $\delta/\sqrt{n}$. Let $J$ be any subbox of $P$. By the extreme value theorem, $\exists\, x_J, y_J \in J$ such that $m_J(f) = f(x_J)$ and $M_J(f) = f(y_J)$. Hence,

$$M_J(f) - m_J(f) = f(y_J) - f(x_J) < \frac{\varepsilon}{\text{vol}(B)}$$

since $|x_J - y_J| < \delta$. It follows that

$$
\begin{aligned}
U(f,P) - L(f,P) &= \sum_J{}' M_J(f)\,\mathrm{vol}(J) - \sum_J{}' m_J(f)\,\mathrm{vol}(J) \\
&= \sum_J{}' \left[ M_J(f) - m_J(f) \right] \mathrm{vol}(J) \\
&< \sum_J{}' \frac{\varepsilon}{\mathrm{vol}(B)}\,\mathrm{vol}(J) = \frac{\varepsilon}{\mathrm{vol}(B)} \sum_J{}' \mathrm{vol}(J) \\
&= \frac{\varepsilon}{\mathrm{vol}(B)} \cdot \mathrm{vol}(B) = \varepsilon.
\end{aligned}
$$

So given $\varepsilon > 0$, $\exists P$ s.t. $U(f,P) - L(f,P) < \varepsilon$. Therefore, $f$ is integrable by the integrability criterion.

Next goal: recall a proof of the fundamental theorem of calculus in 1-dimension.

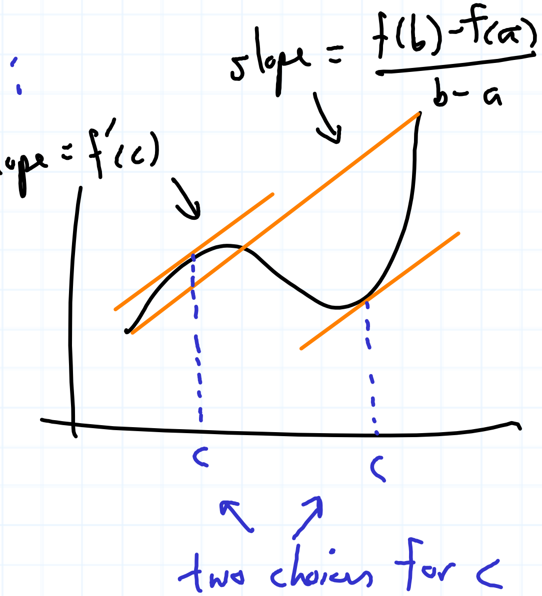
Thm. (Mean Value Thm.) Let $f : [a,b] \to \mathbb{R}$ with $f$ continuous on $[a,b]$ and differentiable on $(a,b)$. Then $\exists\, c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

instantaneous speed

average speed

Picture:




Pf/ See a 1-variable calculus text. It follows from Rolle's thm. $\square$

**Thm. (FTC in 1-variable)**

Suppose $f : [a,b] \to \mathbb{R}$ is integrable, and suppose there is a function $g : [a,b] \to \mathbb{R}$, continuous on $[a,b]$ and differentiable on $(a,b)$ such that $g' = f$. Then $\int_a^b f = g(b) - g(a)$.

Pf/ Given a partition $P: a = t_0 < t_1 < \cdots < t_n = b$, apply the MVT to $g$ on each subinterval to get $c_i \in [t_{i-1}, t_i]$ s.t.

$$f(c_i) \underset{\uparrow}{=} g'(c_i) = \frac{g(t_i)-g(t_{i-1})}{t_i - t_{i-1}}.$$

since $f=g'$

Therefore, $f(c_i)(t_i - t_{i-1}) = g(t_i) - g(t_{i-1}) \quad \forall i.$ $\quad (\star)$

For each subinterval $J_i = [t_{i-1}, t_i]$, we have

$$m_J(f) \le f(c_i) \le M_J(f) \implies$$

$$L(f,P) = \sum_{i=1}^n m_{J_i}(f)(t_i - t_{i-1}) \le \sum_{i=1}^n \underbrace{f(c_i)(t_i - t_{i-1})}_{= g(t_i) - g(t_{i-1}) \text{ by } (\star)} \le \sum_{n=1}^n M_{J_i}(f)(t_i - t_{i-1}) = U(f,P)$$

$$\implies L(f,P) = \sum_{i=1}^n (g(t_i) - g(t_{i-1})) \le U(f,P)$$

But $\sum_{i=1}^n (g(t_i) - g(t_{i-1})) = g(t_n) - g(t_0) = g(b) - g(a).$

So $\quad L(f,P) \le g(b) - g(a) \le U(f,P)$

which implies $\quad \underline{\int} f = \sup_P \{L(f,P)\} \le g(b) - g(a) \le \inf_P \{U(f,P)\} = \overline{\int} f.$

But $\underline{\int} f = \overline{\int} f$, and the result follows. $\square$