Math 212

Next goal: f continuous $\Rightarrow$ f integrable.

Example of a non-integrable function

Let $f : [0,1] \to \mathbb{R}$

$$x \mapsto \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x \text{ is rational}. \end{cases}$$

Let P be any partition of $[0,1]$. Then for any subinterval J of P, we have $m_J(f) = 0$ and $M_J(f) = 1$. Hence,

$$L(f,P) = \sum_J m_J(f)\,\mathrm{vol}(J) = 0 \quad \text{and} \quad U(f,P) = \sum_J M_J(f)\,\mathrm{vol}(J)$$

$$= \sum \mathrm{vol}(J) = \mathrm{len}([0,1]) = 1.$$

Since, P is arbitrary, this says

$$\{L(f,P) : P \text{ partition of } [0,1]\} = \{0\} \quad \text{and} \quad \{U(f,P) : P \text{ part.}\} = \{1\}.$$

So $\underline{\int} f = \sup_P \{L(f,P)\} = \sup \{0\} = 0$ and

$\overline{\int} f = \sup_P \{U(f,P)\} = \sup \{1\} = 1$. Since $\underline{\int} f \neq \overline{\int} f$,

the function f is not integrable. $\square$

Lemma (Integrability criterion). Let B be a box in $\mathbb{R}^n$, and let $f: B \to \mathbb{R}$ be a bounded function. Then f is integrable iff $\forall \varepsilon > 0$, $\exists$ partition P of B such that

$$U(f,P) - L(f,P) < \varepsilon.$$

Pf/ ($\Rightarrow$) First suppose f is integrable, i.e.

$$\underline{\int} f = \sup_P \{L(f,P)\} = \inf_P \{U(f,P)\} = \overline{\int} f.$$

Given $\varepsilon > 0$, since $\underline{\int} f - \frac{\varepsilon}{2} < \sup_P \{L(f,P)\}$,

$\exists$ partition P' s.t. $\underline{\int} f - \frac{\varepsilon}{2} < L(f,P')$. Similarly, $\exists$ partition P'' s.t. $\overline{\int} f + \frac{\varepsilon}{2} > U(f,P'')$. Letting P be the common refinement of P' and P'', and recalling that $\underline{\int} f = \overline{\int} f = \int f$, we have

$$\underbrace{\int f - \frac{\varepsilon}{2} < L(f,P') \leq L(f,P) \leq U(f,P) \leq U(f,P'') < \int f + \frac{\varepsilon}{2}}_{\text{distance is } < \varepsilon}.$$

It follows that $U(f,P) - L(f,P) < \varepsilon$.

($\Leftarrow$) Suppose that given $\varepsilon > 0$, $\exists$ P s.t. $U(f,P) - L(f,P) < \varepsilon$. Since $L(f,P) \leq \underline{\int} f \leq \overline{\int} f \leq U(f,P)$, we would then have

$$\left| \overline{\int} f - \underline{\int} f \right| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary and $\underline{\int} f$ and $\overline{\int} f$ are fixed real numbers (not dependent on P), it must be that $\left| \overline{\int} f - \underline{\int} f \right| = 0$, i.e. $\underline{\int} f = \overline{\int} f$, i.e. f is integrable. $\square$

The key idea behind proving f cts. $\Rightarrow f$ integrable is uniform continuity.

$\star$ δ is a function of ε <u>and</u> <u>of</u> <u>s</u>

Def. Let $S \subseteq \mathbb{R}^n$ and let $f: S \to \mathbb{R}$. Then f is **continuous at $s \in S$** if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $|x - s| < \delta$ (and $x \in \text{domain}(f)$) $\Rightarrow |f(x) - f(s)| < \varepsilon$. The function f is **uniformly continuous on S** if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.

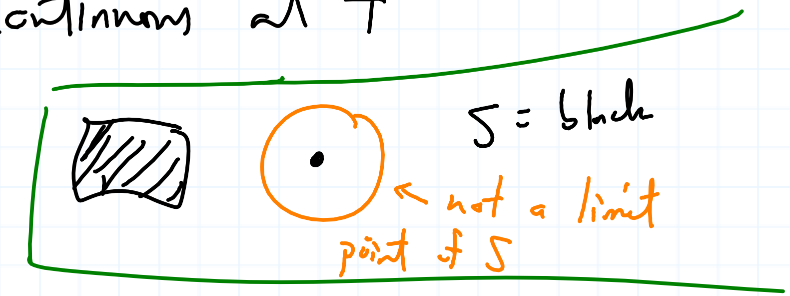
$\star$ δ is only a function of ε

Remarks

Every nonempty open set containing s contains a point of S besides s.

* If s is a limit point of S, then f is continuous at f iff $\lim\limits_{x \to s} f(x) = f(s)$ [lim "commutes" with f]. If s is not a limit point of S, then f is automatically continuous at f

* $f: (0, \infty) \to \mathbb{R}$, $x \mapsto \frac{1}{x}$ is continuous but not uniformly.



Reason: The function gets very steep as $x \to 0$. So in order for $|f(x) - f(y)|$ to be small, we need x, y closer and closer together as x and y get close to 0.

Proof f is not uniformly continuous / Let $\varepsilon = 1$ and take any $\delta > 0$. Define $x_n = \frac{1}{n+1}$ and $y_n = \frac{1}{n}$ for each $n \in \mathbb{Z}_{>0}$. Then $|x_n - y_n| = \frac{1}{n(n+1)}$. Pick $n >> 0$ so that $|x_n - y_n| < \delta$.

Then $|x_n - y_n| < \delta$ but $|f(x_n) - f(y_n)| = |n - (n+1)| = 1 \not< \varepsilon = 1$. $\square$

Thm. Let $K \subseteq \mathbb{R}^n$ be compact (closed and bounded), and let $f: K \to \mathbb{R}$ be continuous. Then f is uniformly continuous.

Pf/ Next time. $\square$