Math 212

Infs + Sups (Review)

Def. Let $X \subseteq \mathbb{R}$. Then $s \in \mathbb{R}$ is the infimum of X if it is the greatest lower bound for X:

(i) $s \leq x \quad \forall x \in X$ [s is a lower bound]

(ii) if $s' \leq x \;\; \forall x \in X$, then $s' \leq s$. [s is at least as large as all other lower bounds]

Similarly $t \in \mathbb{R}$ is the supremum of X if it is the least upper bound for X:

(i) $t \geq x \quad \forall x \in X$

(ii) if $t' \geq x \;\; \forall x \in X$, then $t' \geq t$.

Thm. $\mathbb{R}$ is complete: every nonempty subset of $\mathbb{R}$ that is bounded above has a supremum.

Exercises

1. Give an example of a set of real numbers that is bounded above but has no supremum.

Solution: The empty set $\emptyset$. Every real number is an upper bound for $\emptyset$: if $t \in \mathbb{R}$, then it is true that $t \geq x \ \forall x \in \emptyset$. $\square$

Maybe exercise 5 should go here.

2. Prove that if $\emptyset \neq X \subseteq Y \subseteq \mathbb{R}$ and $\tilde{y} = \sup Y$, then $\tilde{x} = \sup X$ exists and $\tilde{x} \leq \tilde{y}$. (There is a similar result for infs.)

Solution: The set X is bounded above by $\tilde{y}$: $x \in X \Rightarrow x \in Y \Rightarrow x \leq \tilde{y}$. By completeness, $\tilde{x} = \sup X$ exists.

Then, since $\tilde{y}$ is an upper bound for X and $\tilde{x}$ is the least upper bound for X, we have $\tilde{x} \leq \tilde{y}$.

3. Prove that if the supremum of a set $X \subseteq \mathbb{R}$ exists, it is unique.

Solution: Let t, t' both be supremums for X. Then since t is an upper bound for X and t' is a least upper bound, we have $t' \leq t$. Similarly, $t \leq t'$. Hence, $t = t'$.

4. Let s be an upper bound for $X \subseteq \mathbb{R}$. Show that
$$s = \sup X \text{ iff } \forall \varepsilon > 0, \exists x \in X \text{ s.t. } 0 \leq s - x < \varepsilon.$$

Solution: ($\Rightarrow$) Suppose $s = \sup X$ and let $\varepsilon > 0$. Since $s - \varepsilon < s$ and s is the least upper bound for X, it follows that $s - \varepsilon$ is not an upper bound for X. Hence, $\exists x \in X$

$s - \varepsilon < x$, whence $s - x < \varepsilon$ (and $0 \leq s - x$ since s is an upper bound for X).

($\Leftarrow$) Now suppose that $\forall \varepsilon > 0, \exists x \in X$ such that $0 \leq s - x < \varepsilon$. Suppose $s' < s$. We need to show that s' is not an upper bound for X. Let $\varepsilon = s - s'$. By supposition, $\exists x \in X$ s.t. $0 \leq s - x < \varepsilon = (s - s')$. Then $s - x < s - s' \Rightarrow x > s'$, so s' is not an upper bound for X.

5. Prove that if $\inf X$ and $\sup X$ exist, then $\inf X \leq \sup X$.

Solution: Suppose $\inf X$ and $\sup X$ exist. It follows that $X \neq \emptyset$ (see exercise 1). So let $x \in X$. It follows that $\inf X \leq x$ since $\inf X$ is a lower bound for X. Similarly,

$x \leq \sup X$. Therefore $\inf X \leq x \leq \sup X$.

6. If $X \subseteq \mathbb{R}$, define $-X = \{-x : x \in X\}$. If $\tilde{x} = \sup X$, prove that $-\tilde{x} = \inf(-X)$.

Solution: Let $y \in -X$. Then $-y \in X$, so $-y \leq \tilde{x}$. It follows that $y \geq -\tilde{x}$. Hence $-\tilde{x}$ is a lower bound for $-X$. Suppose $t \leq y$ $\forall y \in -X$. Let $x \in X$. Then $-x \in -X \Rightarrow$ $t \leq -x \Rightarrow -t \geq x \Rightarrow -t$ is an upper bound for X.

Therefore, $-t \geq \tilde{x}$ since $\tilde{x}$ is the least upper bound. Hence, $t \leq -\tilde{x}$. We've shown $-\tilde{x}$ is at least as large as all lower bounds t for $-X$. Hence, $-\tilde{x} = \inf(-X)$.

7. Show that if X is a nonempty subset of $\mathbb{R}$ bounded below, then X has an infimum.

Solution: Let b be a lower bound for X. Then $-b$ is an upper bound for $-X$. By completeness of $\mathbb{R}$, $\tilde{x} = \sup(-X)$ exists. By the previous exercise, $-\tilde{x} = \inf(-(-X)) = \inf(X)$.